

Production of B_c or \bar{B}_c meson and its excited states via \bar{t} -quark or t -quark decays

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Abstract

The production of $(b\bar{c})$ -quarkonium (\bar{B}_c meson and its excited states) or $(c\bar{b})$ -quarkonium (B_c meson and its excited states) via top quark t or top anti-quark \bar{t} decays, $t \rightarrow (b\bar{c}) + c + W^+$ or $\bar{t} \rightarrow (c\bar{b}) + \bar{c} + W^-$, respectively is studied within the framework of NRQCD. In addition to the production of the two color-singlet S -wave states $|(b\bar{c})(^1S_0)_1\rangle$ or $|(c\bar{b})(^1S_0)_1\rangle$ and $|(b\bar{c})(^3S_1)_1\rangle$ or $|(c\bar{b})(^3S_1)_1\rangle$, the production of the P -wave excited $(b\bar{c})$ or $(c\bar{b})$ states, i.e. the four color-singlet P -wave states $|(b\bar{c})(^1P_1)_1\rangle$ or $|(c\bar{b})(^1P_1)_1\rangle$, and $|(b\bar{c})(^3P_J)_1\rangle$ or $|(c\bar{b})(^3P_J)_1\rangle$ (with $J = (1, 2, 3)$), is also studied. According to the velocity scaling rule of NRQCD, for the production of P -wave excited states the contributions from the two color-octet components $|(b\bar{c})(^1S_0)_8\rangle$ or $|(c\bar{b})(^1S_0)_8\rangle$ and $|(b\bar{c})(^3S_1)_8\rangle$ or $|(c\bar{b})(^3S_1)_8\rangle$ are also taken into account. We quantitatively discuss the possibility and the advantages in experimental studies of B_c or \bar{B}_c meson and its excited states via the indirect production at LHC in high luminosity runs and at LHC possible upgraded versions such as SLHC, DLHC, TLHC etc. in future.

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I. INTRODUCTION

Since B_c meson was observed by CDF in 1998 [1], the interests in studying of the meson B_c are increasing and a new stage of the studies has been started. As pointed by the authors of Refs.[2, 3], the product cross-section at Large Hadron Collider (LHC), CERN, is much greater than that at Tevatron, Fermilab, therefore, more precise experimental studies of B_c meson are expected at the forthcoming of LHC. Indeed, some progress at LHC in the experimental study of B_c physics can be achieved, especially at the beginning stage of LHC (LHC just runs at 'low' luminosity $L = 10^{32} \sim 10^{33} cm^{-2}s^{-1}$). However, when LHC runs at higher luminosity, such as the design luminosity $L = 10^{34} cm^{-2}s^{-1}$ later on, due to the requests coming from the main purposes of LHC (such as searching for Higgs particle and SUSY partners etc) and the restricts from the abilities to record the events for the detector (e.g. the detector cannot record too frequently events etc), the condition for triggering the events occurring in collisions has to be set such that too many B_c events via the direct hadronic production according to the theoretical estimate of 'direct' B_c -production[2, 3, 4, 5, 6] are cut off. As a result, the direct hadronic production of B_c cannot be expected to make much progress in B_c -meson study in the high luminosity stage of LHC runs. Baring the situation pointed out here and the possible upgrade for LHC (SLHC, DLHC and TLHC etc[7]) in mind, the possibility to study B_c meson experimentally via indirect production of B_c meson, namely, via producing a huge amount of top antiquarks \bar{t} and their decays, is worthwhile to think seriously about. It is because that at LHC no matter how high the luminosity, the produced top quark(s) shall never be cut off by the trigger condition set down for any experimental purposes, and the frequency of the t -quark production can be stood up for the detector always. Furthermore, the mechanisms for producing the doubly heavy mesons, such as η_c , η_b , B_c , \dots and J/ψ , Υ , B_c^* , \dots , are interesting too. The indirect production of B_c or \bar{B}_c (or B_c^-), including its excited states, via \bar{t} -decays or t -decays may offer some knowledge on the mechanisms, therefore, this paper is devoted to study the indirect production of B_c or \bar{B}_c meson via \bar{t} -decays or t -decays. Without confusing and for simplifying the statements, later on we will not distinguish B_c and \bar{B}_c unless necessary, and all results for B_c and \bar{B}_c obtained in the paper are symmetric in the interchange from particle to anti-particle.

The doubly heavy meson production via top quark decays is special interesting from the

point view of precise testing perturbative quantum chromodynamics (pQCD)[8]. The meson \bar{B}_c is the ground state of the heavy-flavored binding system $(b\bar{c})$, and it is unique ‘doubly heavy-flavored’ anti-meson in Standard Model and is stable for strong and electromagnetic interactions. Although in literature the ‘direct’ hadronic production of \bar{B}_c meson has been thoroughly studied, e.g. see Refs. [2, 3, 4, 5, 6] (references therein) and CDF discovered the meson which just come from the ‘direct’ production, as a compensation to understand the production mechanisms, it is quite interesting that to study the production of \bar{B}_c meson ‘indirectly’ through t -quark decays, especially, considering the fact that numerous t -quarks may be produced at LHC. The theoretical studies of the direct production[2, 3, 4, 5, 6] is based on NRQCD [9], so now we study the indirect production based on NRQCD too.

In the framework of effective theory of NRQCD, a doubly heavy meson is considered as an expansion of a series Fock states. The relative importance among the infinite ingredients is accounted by the velocity scaling rule. Namely the physical state of \bar{B}_c , \bar{B}_c^* , $h_{\bar{B}_c}$ and $\chi_{\bar{B}_c}^J$ can be decomposed into a series of Fock states as follows:

$$\begin{aligned} |\bar{B}_c\rangle &= \mathcal{O}(v^0)|(b\bar{c})_{\mathbf{1}}(^1S_0)\rangle + \mathcal{O}(v^2)|(b\bar{c})_{\mathbf{8}}(^1P_1)g\rangle + \cdots \\ |\bar{B}_c^*\rangle &= \mathcal{O}(v^0)|(b\bar{c})_{\mathbf{1}}(^3S_1)\rangle + \mathcal{O}(v^2)|(b\bar{c})_{\mathbf{8}}(^3P_J)g\rangle + \cdots \end{aligned} \quad (1)$$

and

$$\begin{aligned} |h_{\bar{B}_c}\rangle &= \mathcal{O}(v^0)|(b\bar{c})_{\mathbf{1}}(^1P_1)\rangle + \mathcal{O}(v^1)|(b\bar{c})_{\mathbf{8}}(^1S_0)g\rangle + \cdots \\ |\chi_{\bar{B}_c}^J\rangle &= \mathcal{O}(v^0)|(b\bar{c})_{\mathbf{1}}(^3P_J)\rangle + \mathcal{O}(v^1)|(b\bar{c})_{\mathbf{8}}(^3S_1)g\rangle + \cdots, \end{aligned} \quad (2)$$

here v is the relative velocity between the components. The thickened subscripts of the $(b\bar{c})$ denote for color indices, $\mathbf{1}$ for color singlet and $\mathbf{8}$ for color-octet; the relevant angular momentum quantum numbers are shown in the parentheses accordingly. According to the velocity scaling rule of NRQCD, the probability of each Fock state in the expansion is proportional to a definite power in v as indicated as that in Eqs.(1,2). Since the value of v^2 is around $0.1 \sim 0.3$ [9], which is not too small, and the contributions from the two color-octet S-wave components to the P -wave production might be comparable with those from the color-singlet components. So we shall consider the two color-octet components: $|(b\bar{c})_{\mathbf{8}}(^1S_0)g\rangle$ and $|(b\bar{c})_{\mathbf{8}}(^3S_1)g\rangle$, in addition to those color-singlet components in the mesons \bar{B}_c , \bar{B}_c^* , $h_{\bar{B}_c}$ and $\chi_{\bar{B}_c}^J$.

The calculations of the process are very complicated and lengthy by using the conventional trace techniques to calculate the amplitude square due to the two massive particles and bound state effects. To shorten the calculations and to make the results more compact, we adopt the method used in Ref.[10] to do the calculations. For convenience, we will call it as the ‘new trace amplitude approach’. Under this approach, we first arrange the whole amplitude into several orthogonal sub-amplitudes $M_{ss'}$ according to the spins of the t -quark (s') and c -quark (s), and then do the trace of the Dirac γ matrix strings at the amplitude level, which result in explicit series over some independent Lorentz-structures, and finally, we obtain the square of the amplitude. During the calculating, some useful tricks have also been introduced to make the expressions more compact. More detail of the techniques and all the necessary expressions for the amplitudes of $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$ with $(b\bar{c})$ -quarkonium in $|(b\bar{c})(^1S_0)_1\rangle$, $|(b\bar{c})(^3S_1)_1\rangle$, $|(b\bar{c})(^1P_1)_1\rangle$ and $|(b\bar{c})(^3P_J)_1\rangle$ (with $J = (1, 2, 3)$) respectively are put in the APPENDIX B. While the expressions for the contributions from the two color-octet components $|(b\bar{c})(^1S_0)_8\rangle$ and $|(b\bar{c})(^3S_1)_8\rangle$ can be obtained from those from the two color singlet S -wave components by changing the overall color-factor and the corresponding matrix elements. This approach is different from that of the spinor techniques or the so called ‘helicity amplitude approach’, which has been proposed in Ref.[11] and improved in Ref.[12]. Under the ‘helicity amplitude approach’, one can also derive compact results that can be further expressed by spinor-products at the amplitude level and then do the numerical calculations ¹. The above two methods are complement to each other and both can derive simple and compact expressions at the amplitude level. The ‘helicity amplitude approach’ is more suitable for the numerical calculations and is more quicker for calculations, since full components of the helicity amplitude can be evaluated at the amplitude level. While for the ‘new trace amplitude approach’, at the amplitude level only the coefficients of the basic Lorentz structures are numerical. However from the amplitude derived from the ‘new trace amplitude approach’, one can sequentially result in the squared amplitude, which is more easier to be compared with the results derived by the traditional trace techniques.

According to Refs.[13], one may expect at LHC to produce $\sim 10^8$ $t\bar{t}$ -pairs per year

¹ Here, we refer to Ref.[5] for an example on the ‘helicity amplitude approach’, where full processes of the approach from the formulae deduction to the numerical calculation can be seen explicitly. More over some tricks to simplify the massive amplitudes can be found there.

under the luminosity $L = 10^{34} \text{cm}^{-2} \text{s}^{-1}$. Considering the possible upgrade for LHC and the estimate by Refs.[13], we will assume that one may obtain $\sim 10^8 - 10^{10}$ $t\bar{t}$ -pairs per year to examine the possibility to observe the meson B_c via decay of the produced $t\bar{t}$ -pairs precisely and to examine the advantages in the indirect way to observe the meson B_c and its excited states.

The paper is organized as follows. In Sec.II, we show our calculation techniques for the process $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$. Then we present numerical results and make some discussions on the properties of the $(b\bar{c})$ -production through t -quark decays in Sec.III. The fourth section is reserved for a summary. All necessary expressions are put in the appendices finally.

II. CALCULATION TECHNIQUES

Under the NRQCD framework [9, 14], the total decay width for the production of $(b\bar{c})$ -quarkonium through the channel $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$ takes the form:

$$\Gamma = \sum_n H_n(t \rightarrow (b\bar{c}) + c + W^+) \times \frac{\langle \mathcal{O}_n \rangle}{N_{col}}, \quad (3)$$

where N_{col} refers to the number of colors, n stands for the involved state of $b\bar{c}$ -quarkonium. $N_{col} = 1$ for singlets and $N_{col} = N_c^2 - 1$ for octets. With the help of the saturation approximation [9], the involved decay matrix elements can be written as ²,

$$\langle b\bar{c}(^1S_0)_1 | \mathcal{O}_1(^1S_0) | b\bar{c}(^1S_0)_1 \rangle = \left| \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \psi_b | b\bar{c}(^1S_0)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (4)$$

$$\langle b\bar{c}(^3S_1)_1 | \mathcal{O}_1(^3S_1) | b\bar{c}(^3S_1)_1 \rangle = \left| \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma \psi_b | b\bar{c}(^3S_1)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (5)$$

$$\langle b\bar{c}(^1P_1)_1 | \mathcal{O}_1(^1P_1) | b\bar{c}(^1P_1)_1 \rangle = \left| \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}}\right) \psi_b | b\bar{c}(^1P_1)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (6)$$

$$\langle b\bar{c}(^3P_0)_1 | \mathcal{O}_1(^3P_0) | b\bar{c}(^3P_0)_1 \rangle = \left| \frac{1}{\sqrt{3}\sqrt{2N_c}} \langle 0 | \chi_c^+ \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \sigma\right) \psi_b | b\bar{c}(^3P_0)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (7)$$

$$\langle b\bar{c}(^3P_1)_1 | \mathcal{O}(^3P_1)_1 | b\bar{c}(^3P_1)_1 \rangle = \left| \frac{1}{\sqrt{2}\sqrt{2N_c}} \langle 0 | \chi_c^+ \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \times \sigma\right) \psi_b | b\bar{c}(^3P_1)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (8)$$

$$\langle b\bar{c}(^3P_2)_1 | \mathcal{O}_1(^3P_2) | b\bar{c}(^3P_2)_1 \rangle = \left| \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \left(-\frac{i}{2} \overleftrightarrow{D}^{(i)} \sigma^{(j)}\right) \psi_b | b\bar{c}(^3P_2)_1 \rangle \right|^2 [1 + \mathcal{O}(v^4)], \quad (9)$$

where the subscript 1 or 8 indicates that the operator is a color singlet or a color octet, ψ^+ is the Pauli-spinor field that create a heavy quark, χ is the Pauli-spinor that create

² Here as suggested in Ref.[14], an overall factor $1/2N_c$ is introduced into the color-singlet matrix elements.

a heavy antiquark, $D^\mu = \partial^\mu + igA^\mu$ is the gauge-covariant derivative, A is the $SU(3)$ -matrix-valued gauge field. The operator $\vec{\mathbf{D}}$ is the difference between the covariant derivative acting on the spinor to the right and on the spinor to the left, which is defined by $\chi^\dagger \vec{\mathbf{D}} \psi = \chi^\dagger (\mathbf{D} \psi) - (\mathbf{D} \chi)^\dagger \psi$. The notation $T^{(ij)}$ is for the symmetric traceless component of a tensor: $T^{(ij)} = (T^{ij} + T^{ji})/2 - T^{kk} \delta^{ij}$. Furthermore, we have

$$\frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \psi_b | b\bar{c}({}^1S_0)_1 \rangle = \frac{1}{\sqrt{4\pi}} \bar{R}_S(\Lambda) [1 + \mathcal{O}(v^2)] \quad (10)$$

$$\frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma \psi_b | b\bar{c}({}^3S_1)_1(\epsilon) \rangle = \frac{1}{\sqrt{4\pi}} \bar{R}_S(\Lambda) \epsilon [1 + \mathcal{O}(v^2)] \quad (11)$$

$$\frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma (-\frac{i}{2} \vec{\mathbf{D}}) \psi_b | b\bar{c}({}^1P_1)_1(\epsilon) \rangle = \sqrt{\frac{3}{4\pi}} \bar{R}'_P(\Lambda) \epsilon [1 + \mathcal{O}(v^2)] \quad (12)$$

$$\frac{1}{\sqrt{3}} \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma (\frac{1}{2} \vec{\mathbf{D}} \cdot \sigma) \psi_b | b\bar{c}({}^3P_0)_1 \rangle = \sqrt{\frac{3}{4\pi}} \bar{R}'_P(\Lambda) [1 + \mathcal{O}(v^2)] \quad (13)$$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma (-\frac{i}{2} \vec{\mathbf{D}} \times \sigma) \psi_b | b\bar{c}({}^3P_1)_1(\epsilon) \rangle = \sqrt{\frac{3}{4\pi}} \bar{R}'_P(\Lambda) \epsilon [1 + \mathcal{O}(v^2)] \quad (14)$$

$$\frac{1}{\sqrt{2N_c}} \langle 0 | \chi_c^+ \sigma (\frac{1}{2} \vec{D}^{(i)} \sigma^{(j)}) \psi_b | b\bar{c}({}^3P_2)_1(\epsilon) \rangle = \sqrt{\frac{3}{4\pi}} \bar{R}'_P(\Lambda) \epsilon^{ij} [1 + \mathcal{O}(v^2)], \quad (15)$$

where $\bar{R}_S(\Lambda)$ is the average radial wavefunction for $1S$ state averaged over a region of size $1/\Lambda$ centered at origin, $\bar{R}'_P(\Lambda)$ is the average derivative of the radial wavefunction at origin of size $1/\Lambda$. For convenience, we shall take $\bar{R}_S(\Lambda)$ and $\bar{R}'_P(\Lambda)$ to be the phenomenological values $R_S(0)$ and $R'_P(0)$, which may be derived from the QCD potential models and relate to certain observable such as the width for electromagnetic annihilation etc.

Although we do not know the exact values of the two decay color-octet matrix elements, $\langle b\bar{c}({}^1S_0)_8 | \mathcal{O}_8({}^1S_0) | b\bar{c}({}^1S_0)_8 \rangle$ and $\langle b\bar{c}({}^3S_1)_8 | \mathcal{O}_8({}^3S_1) | b\bar{c}({}^3S_1)_8 \rangle$, we know that they are one order in v^2 higher than the S -wave color-singlet matrix elements according to NRQCD scale rule. More specifically, based on the velocity scale rule, we have:

$$\begin{aligned} \langle b\bar{c}({}^1S_0)_8 | \mathcal{O}_8({}^1S_0) | b\bar{c}({}^1S_0)_8 \rangle &\simeq \Delta_S(v)^2 \cdot \langle b\bar{c}({}^1S_0)_1 | \mathcal{O}_1({}^1S_0) | b\bar{c}({}^1S_0)_1 \rangle \\ \langle b\bar{c}({}^3S_1)_8 | \mathcal{O}_8({}^3S_1) | b\bar{c}({}^3S_1)_8 \rangle &\simeq \Delta_S(v)^2 \cdot \langle b\bar{c}({}^3S_1)_1 | \mathcal{O}_1({}^3S_1) | b\bar{c}({}^3S_1)_1 \rangle, \end{aligned} \quad (16)$$

where the second equation comes from the vacuum-saturation approximation. $\Delta_S(v)$ is of the order v^2 or so, and we take it to be within the region of 0.10 to 0.30, which is in consistent with the identification: $\Delta_S(v) \sim \alpha_s(Mv)$ and has covered the possible variation due to the different ways to obtain the wave functions at the origin (S -wave) and the first derivative of the wave functions at the origin (P -wave) etc.

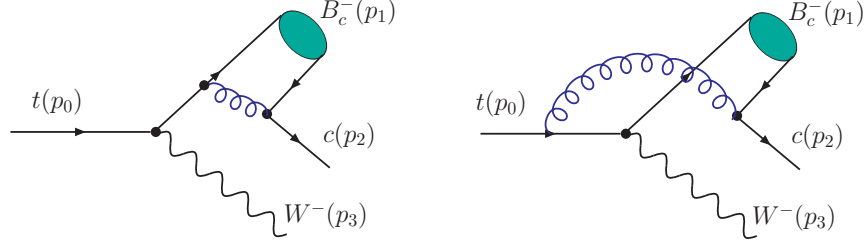


FIG. 1: Feynman diagrams for $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$, where $b\bar{c}$ -quarkonium is in eight states, i.e. the six color-singlet states $(^1S_0)_1$, $(^3S_1)_1$, $(^1P_1)_1$ and $(^3P_J)_1$ (with $J = (1, 2, 3)$), and two color-octet states $(^1S_0)_8$ and $(^3S_1)_8$ respectively.. The left and the right diagram are for the hard scattering amplitudes \mathcal{A}_1 and \mathcal{A}_2 respectively.

The Feynman diagrams for the process of $t(p_0) \rightarrow b\bar{c}(p_1) + c(p_2) + W^+(p_3)$ are shown in Fig.(1), where $b\bar{c}$ -quarkonium is in eight states: six color-singlet components $(^1S_0)_1$, $(^3S_1)_1$, $(^1P_1)_1$ and $(^3P_J)_1$ (with $J = (1, 2, 3)$), and two color-octet components $(^1S_0)_8$ and $(^3S_1)_8$ respectively. Based on the phase-space integration simplification as shown in the Appendix A, the decay width of the process can be written in the form:

$$d\Gamma = \frac{3|\bar{M}|^2}{256\pi^3 m_t^3} \times \frac{\langle \mathcal{O}_n \rangle}{N_{col}} ds_1 ds_2, \quad (17)$$

where the extra factor 3 in the numerator comes from the sum of the c -quark color, $|\bar{M}|^2$ is the mean square of the hard scattering amplitude, i.e. $|\bar{M}|^2 = \frac{1}{2 \times 3} \sum |\mathcal{A}_1 + \mathcal{A}_2|^2$ with \mathcal{A}_1 and \mathcal{A}_2 are two amplitudes of the process, $s_1 = (p_1 + p_2)^2$ and $s_2 = (p_2 + p_3)^2$.

We deal with either $L = 0$ or $L = 1$ state here. The two hard scattering amplitudes of the process, which correspond to the left and right diagram of Fig.(1), can be written as

$$\mathcal{A}_1^{S=0, L=0} = i\mathcal{C}\bar{u}_i(p_2, s) \left[\gamma_\mu \frac{\Pi_{p_1}^0(q)}{(p_2 + p_{11})^2} \gamma_\mu \frac{\not{p}_1 + \not{p}_2 + m_b}{(p_1 + p_2)^2 - m_b^2} \not{\epsilon}(p_3) P_L \right] u_j(p_0, s')|_{q=0} \quad (18)$$

$$\mathcal{A}_2^{S=0, L=0} = i\mathcal{C}\bar{u}_i(p_2, s) \left[\gamma_\mu \frac{\Pi_{p_1}^0(q)}{(p_2 + p_{11})^2} \not{\epsilon}(p_3) P_L \frac{\not{p}_{12} + \not{p}_3 + m_t}{(p_{12} + p_3)^2 - m_t^2} \gamma_\mu \right] u_j(p_0, s')|_{q=0} \quad (19)$$

and

$$\mathcal{A}_1^{S=1, L=0} = i\mathcal{C}\bar{u}_i(p_2, s) \left[\gamma_\mu \frac{\varepsilon_s^\alpha(p_1) \Pi_{p_1}^\alpha(q)}{(p_2 + p_{11})^2} \gamma_\mu \frac{\not{p}_1 + \not{p}_2 + m_b}{(p_1 + p_2)^2 - m_b^2} \not{\epsilon}(p_3) P_L \right] u_j(p_0, s')|_{q=0} \quad (20)$$

$$\mathcal{A}_2^{S=1, L=0} = i\mathcal{C}\bar{u}_i(p_2, s) \left[\gamma_\mu \frac{\varepsilon_s^\alpha(p_1) \Pi_{p_1}^\alpha(q)}{(p_2 + p_{11})^2} \not{\epsilon}(p_3) P_L \frac{\not{p}_{12} + \not{p}_3 + m_t}{(p_{12} + p_3)^2 - m_t^2} \gamma_\mu \right] u_j(p_0, s')|_{q=0} \quad (21)$$

and

$$\mathcal{A}_1^{S=0, L=1} = i\mathcal{C}\varepsilon_l^\alpha(p_1) \bar{u}_i(p_2, s) \frac{d}{dq_\alpha} \left[\gamma_\mu \frac{\Pi_{p_1}^0(q)}{(p_2 + p_{11})^2} \gamma_\mu \frac{\not{p}_1 + \not{p}_2 + m_b}{(p_1 + p_2)^2 - m_b^2} \not{\epsilon}(p_3) P_L \right] u_j(p_0, s')|_{q=0} \quad (22)$$

$$\mathcal{A}_2^{S=0,L=1} = i\mathcal{C}\varepsilon_l^\alpha(p_1)\bar{u}_i(p_2, s)\frac{d}{dq_\alpha}\left[\gamma_\mu\frac{\Pi_{p_1}^0(q)}{(p_2+p_{11})^2}\not{\varepsilon}(p_3)P_L\frac{\not{p}_{12}+\not{p}_3+m_t}{(p_{12}+p_3)^2-m_t^2}\gamma_\mu\right]u_j(p_0, s')|_{q=0} \quad (23)$$

and

$$\mathcal{A}_1^{S=1,L=1} = i\mathcal{C}\varepsilon_{\alpha\beta}^J(p_1)\bar{u}_i(p_2, s)\frac{d}{dq_\alpha}\left[\gamma_\mu\frac{\Pi_{p_1}^\beta(q)}{(p_2+p_{11})^2}\gamma_\mu\frac{\not{p}_1+\not{p}_2+m_b}{(p_1+p_2)^2-m_b^2}\not{\varepsilon}(p_3)P_L\right]u_j(p_0, s')|_{q=0} \quad (24)$$

$$\mathcal{A}_2^{S=1,L=1} = i\mathcal{C}\varepsilon_{\alpha\beta}^J(p_1)\bar{u}_i(p_2, s)\frac{d}{dq_\alpha}\left[\gamma_\mu\frac{\Pi_{p_1}^\beta(q)}{(p_2+p_{11})^2}\not{\varepsilon}(p_3)P_L\frac{\not{p}_{12}+\not{p}_3+m_t}{(p_{12}+p_3)^2-m_t^2}\gamma_\mu\right]u_j(p_0, s')|_{q=0} \quad (25)$$

with the color factor $\mathcal{C} = \mathcal{C}_s$ or \mathcal{C}_o for color-singlet and color-octet respectively, $\mathcal{C}_s = \frac{4gg_s^2}{3\sqrt{6}}\delta_{ij}$ and $\mathcal{C}_o = \frac{gg_s^2}{\sqrt{2}}(\sqrt{2}T^a T^b T^a)_{ij}$ ($\sqrt{2}T^b$ stands for the color of the color-octet $b\bar{c}$ state). $\varepsilon(p_3)$ is the polarization vector of W^+ , $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$. q , p_{11} and p_{12} are the relative momentum between the two constitute quarks of $(b\bar{c})$ -quarkonium and the momenta of these two constitute quarks respectively. More explicitly, we have

$$p_{11} = \frac{m_c}{M}p_1 + q \quad \text{and} \quad p_{12} = \frac{m_b}{M}p_2 - q, \quad (26)$$

where $M \simeq m_b + m_c$. $\varepsilon_s^\alpha(p_1)$ and $\varepsilon_l^\alpha(p_1)$ are the polarization vectors relating to the spin and the orbit angular momentum of $(b\bar{c})$ -quarkonium, $\varepsilon_{\alpha\beta}^J(p_1)$ is the polarization tensor for the spin triplet P -wave states with $J = 0, 1$ and 2 respectively. The covariant form of the projectors can be conveniently written as

$$\Pi_{p_1}^0(q) = \frac{-\sqrt{M}}{4m_b m_c}(\not{p}_{11} - m_c)\gamma_5(\not{p}_{12} + m_b), \quad (27)$$

and

$$\Pi_{p_1}^\alpha(q) = \frac{-\sqrt{M}}{4m_b m_c}(\not{p}_{11} - m_c)\gamma^\alpha(\not{p}_{12} + m_b), \quad (28)$$

To do the simplification of the projector, the following simplification shall be useful:

$$\Pi_{p_1}^0(0) = \frac{1}{2\sqrt{M}}\gamma_5(\not{p}_1 + M) \quad , \quad \Pi_{p_1}^\alpha(0) = \frac{1}{2\sqrt{M}}\gamma^\alpha(\not{p}_1 + M), \quad (29)$$

and

$$\frac{d}{dq_\alpha}\Pi_{p_1}^0(q)|_{q=0} = \frac{\sqrt{M}}{4m_b m_c}\gamma_5\gamma_\alpha(\not{p}_1 + m_b - m_c), \quad (30)$$

$$\frac{d}{dq_\alpha}\Pi_{p_1}^\beta(q)|_{q=0} = -\frac{\sqrt{M}}{4m_b m_c}\left[\gamma_\alpha\gamma_\beta(\not{p}_1 + m_b - m_c) - 2g_{\alpha\beta}(\not{p}_{11}^0 - m_c)\right]. \quad (31)$$

Here the properties: $p_1^\alpha\varepsilon^\alpha = 0$ and $p_1^\alpha\varepsilon^{\alpha\beta} = p_1^\beta\varepsilon^{\alpha\beta} = 0$. $p_{11}^0 = \frac{m_c}{M}p_1$ and $p_{12}^0 = \frac{m_b}{M}p_2$, are applied. After substituting all these relations into the amplitudes and doing the possible

simplifications, the amplitudes then be squared, summed over the freedoms in final state and averaged over the ones in initial state. And the selection of the appropriate total angular momentum quantum number is done by performing the proper polarization sum. If defining

$$\Pi_{\alpha\beta} = -g_{\alpha\beta} + \frac{p_{1\alpha}p_{1\beta}}{M^2}, \quad (32)$$

the sum over polarization for a spin triplet S-state or a spin singlet P-state is given by

$$\sum_{J_z} \varepsilon_\alpha \varepsilon_{\alpha'}^* = \Pi_{\alpha\alpha'}, \quad (33)$$

where $J_z = s_z$ or l_z respectively. In the case of 3P_J states, as for the three multiplets $\varepsilon_{\alpha\beta}^J(p_1)$ with $J = 0, 1$ and 2 , the sum over the polarization is given by

$$\varepsilon_{\alpha\beta}^{(0)} \varepsilon_{\alpha'\beta'}^{(0)*} = \frac{1}{3} \Pi_{\alpha\beta} \Pi_{\alpha'\beta'} \quad (34)$$

$$\sum_{J_z} \varepsilon_{\alpha\beta}^{(1)} \varepsilon_{\alpha'\beta'}^{(1)*} = \frac{1}{2} (\Pi_{\alpha\alpha'} \Pi_{\beta\beta'} - \Pi_{\alpha\beta'} \Pi_{\alpha'\beta}) \quad (35)$$

$$\sum_{J_z} \varepsilon_{\alpha\beta}^{(2)} \varepsilon_{\alpha'\beta'}^{(2)*} = \frac{1}{2} (\Pi_{\alpha\alpha'} \Pi_{\beta\beta'} + \Pi_{\alpha\beta'} \Pi_{\alpha'\beta}) - \frac{1}{3} \Pi_{\alpha\beta} \Pi_{\alpha'\beta'}. \quad (36)$$

All the terms for the squared hard scattering amplitudes $|\bar{M}|^2$ for S-wave and P-wave states are put in Appendix B accordingly, where the detail of the calculation techniques are also attached for convenience. To effectuate the calculations and to make the results more compact, we adopt the method used in Ref.[10]. As stated in the Introduction, we call it the ‘new trace amplitude approach’ for convenience. Under the approach, we arrange the whole amplitude into several orthogonal sub-amplitudes $M_{ss'}$ according to the spins of the t -quark (s') and c -quark (s) first, and then do the trace of the Dirac γ matrix strings at the amplitude level by properly dealing with the massive spinors, which result in explicit series over some independent Lorentz-structures. The expressions for these coefficients of all the considered channels are put in Appendix B. With the help, one can do the square of the amplitude easily. As a cross-check of our results, we adopt the traditional trace techniques and also the FDC[15] package to derive the numerical results of the mentioned processes. Indeed we find a well agreement among these methods.

As a comparison and for later usages, let us present the width for the two body decay $t(p_1) \rightarrow b(p_2) + W^+(p_3)$, which is dominant for the t -quark decay:

$$\Gamma = \frac{G_F m_t^2 |\vec{\mathbf{p}}_2|}{4\sqrt{2}\pi} \left[(1-y^2)^2 + x^2(1+y^2-2x^2) \right], \quad (37)$$

where $|\vec{\mathbf{p}}_2| = \frac{m_t}{2} \sqrt{(1-(x-y)^2)(1-(x+y)^2)}$, $m_w = m_t x$ and $m_b = m_t y$.

III. NUMERICAL RESULTS OF DIFFERENTIAL CROSS-SECTIONS

In numerical calculations, we take the parameters as follows:

$$m_b = 4.9 \text{ GeV}, m_c = 1.5 \text{ GeV}, m_t = 176 \text{ GeV}, m_w = 80.22 \text{ GeV}, \alpha_s(2m_c) = 0.26, \quad (38)$$

and $g = 2\sqrt{2}m_w\sqrt{G_F/\sqrt{2}}$. Then, the decay width of $t \rightarrow W^+ + b$ is

$$\Gamma(t \rightarrow W^+ + b) = 1.59 \text{ GeV}. \quad (39)$$

And the decay widths of $t \rightarrow (b\bar{c}) + W^+ + c$ are:

$$\Gamma_{t \rightarrow (b\bar{c})[(^3S_1)_1]} = 0.79 \text{ MeV} \quad (40)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^1S_0)_1]} = 0.57 \text{ MeV} \quad (41)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^1P_1)_1]} = 0.057 \text{ MeV} \quad (42)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^3P_0)_1]} = 0.034 \text{ MeV} \quad (43)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^3P_1)_1]} = 0.070 \text{ MeV} \quad (44)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^3P_2)_1]} = 0.075 \text{ MeV} \quad (45)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^3S_1)_8]} = 0.091 \times v^4 \text{ MeV} \quad (46)$$

$$\Gamma_{t \rightarrow (b\bar{c})[(^1S_0)_8]} = 0.070 \times v^4 \text{ MeV} \quad (47)$$

where $v^2 \simeq (0.1 \sim 0.3)$. It may be more useful to show the ratio between the decay width of $t \rightarrow (b\bar{c}) + W^+ + c$ and $t \rightarrow W^+ + b$, since uncertainties from the electro-weak coupling can be cancelled out:

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^3S_1)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 4.97 \times 10^{-4} \quad (48)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^1S_0)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 3.58 \times 10^{-4} \quad (49)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^1P_1)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 0.36 \times 10^{-4} \quad (50)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^3P_0)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 0.21 \times 10^{-4} \quad (51)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^3P_1)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 0.44 \times 10^{-4} \quad (52)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^3P_2)_1]}}{\Gamma(t \rightarrow W^+ + b)} = 0.47 \times 10^{-4} \quad (53)$$

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^3S_1)_8]}}{\Gamma(t \rightarrow W^+ + b)} = 0.57 \times 10^{-4} v^4 \quad (54)$$

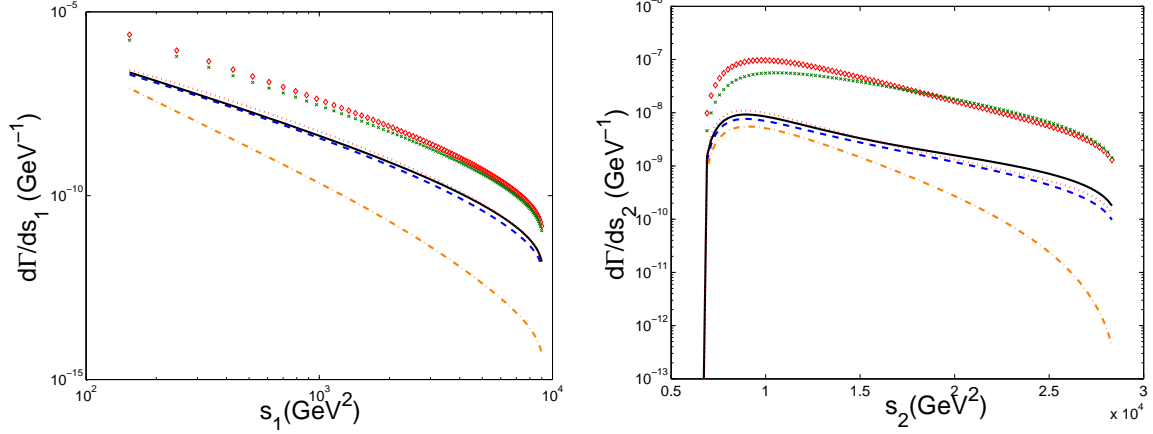


FIG. 2: The invariant mass distributions for $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$. The left is for $d\Gamma/ds_1$ and the right is for $d\Gamma/ds_2$. The diamond line, the cross line, the dotted line, the solid line, the dashed line and the dash-dot line are for $(b\bar{c})$ -quarkonium in Fock states: $(^3S_1)_1$, $(^1S_0)_1$, $(^3P_2)_1$, $(^3P_1)_1$, $(^1P_1)_1$ and $(^3P_0)_1$ respectively.

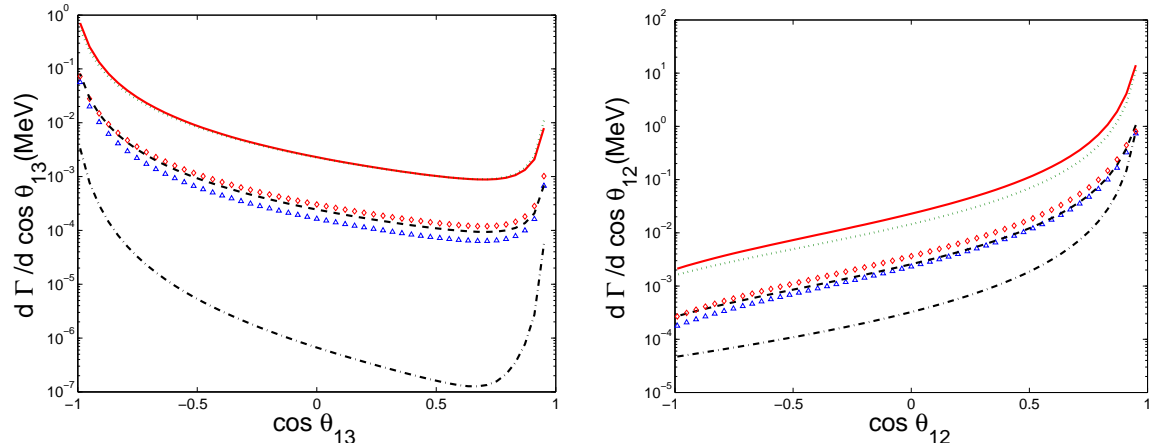


FIG. 3: The differential distributions $d\Gamma/d\cos\theta_{13}$ (Left) and $d\Gamma/d\cos\theta_{12}$ (Right) for $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$. The solid line, the dotted line, the diamond line, the dashed line, the triangle line and the dash-dot line are for $(b\bar{c})$ -quarkonium in Fock states: $(^3S_1)_1$, $(^1S_0)_1$, $(^3P_2)_1$, $(^3P_1)_1$, $(^1P_1)_1$ and $(^3P_0)_1$ respectively.

$$\frac{\Gamma_{t \rightarrow (b\bar{c})[(^1S_0)_8]}}{\Gamma(t \rightarrow W^+ + b)} = 0.44 \times 10^{-4} v^4. \quad (55)$$

Let us show some more characteristics of the decay $t \rightarrow (b\bar{c}) + W^+ + c$. The differential distributions of the invariant masses s_1 and s_2 , i.e. $d\Gamma/ds_1$ and $d\Gamma/ds_2$ are shown in Fig.2. While the differential distributions of $\cos\theta_{13}$ and $\cos\theta_{12}$, i.e. $d\Gamma/d\cos\theta_{13}$ and $d\Gamma/d\cos\theta_{12}$ is

shown in Fig.3, where θ_{13} is the angle between \vec{p}_1 and \vec{p}_3 , and θ_{12} is the angle between \vec{p}_1 and \vec{p}_2 respectively in the t -quark rest frame ($\vec{p}_0 = 0$). It can be found that the largest differential cross-section of $d\Gamma/d\cos\theta_{13}$ is achieved when $\theta_{13} = 180^\circ$, i.e. the $(b\bar{c})$ -quarkonium and W^+ moving back to back in the rest frame of t -quark. And the largest differential cross-section of $d\Gamma/d\cos\theta_{12}$ is achieved when $\theta_{12} = 0^\circ$, i.e. the $(b\bar{c})$ -quarkonium and c -quark moving in the same direction.

IV. DISCUSSIONS AND SUMMARY

In the present paper, we have studied the decay channel $t(p_0) \rightarrow b\bar{c}(p_1) + c(p_2) + W^+(p_3)$ in the leading α_s calculation but with the v^2 -expansion up to v^4 , where $b\bar{c}$ -quarkonium is in one of the eight states: the six color-singlet states $(^1S_0)_1$, $(^3S_1)_1$, $(^1P_1)_1$ and $(^3P_J)_1$ (with $J = (1, 2, 3)$), and two color-octet states $(^1S_0)_8$ and $(^3S_1)_8$ respectively. In literature, only $(^3S_1)_1$ state has been studied [8], however it can be found that all the other considered states can be sizable in addition to the $(^3S_1)_1$ state.

As mentioned in the Introduction, about 10^8 $t\bar{t}$ per year will be produced in the stage of high luminosity run at LHC and t -quark events always trigger the detector, then according to the present estimate it is possible to accumulate about 10^5 B_c events a year via \bar{t} -quark decay at LHC. Moreover, the indirect production of the $(b\bar{c})$ -quarkonium may be traced back to t -quark decay and has the characteristics in θ_{13}, θ_{12} shown in Figs.(2,3) etc, that may be used to identify the $(b\bar{c})$ -quarkonium events. Thus there may be some advantages in $(b\bar{c})$ -quarkonium studies via the indirect production in comparison with the direct production. Especially when LHC is really upgraded to SLHC, DLHC and TLHC, so $10^9 \sim 10^{10}$ $t\bar{t}$ per year may be produced, one may expect much progress in $(b\bar{c})$ -quarkonium studies is achieved then.

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APPENDIX A: FORMULAE FOR THE PHASE SPACE INTEGRATION

In this section, we derive the phase space of $t(m_t, p_0) \rightarrow (b\bar{c})(m_1, p_1) + c(m_2, p_2) + W^+(m_3, p_3)$. The decay width is proportional to the phase space:

$$d\Gamma \propto \frac{1}{2p_0^0} \prod_{i=1}^3 \frac{d^3 p_i}{(2\pi)^3 (2p_i^0)} (2\pi)^4 \delta^4(p_0 - p_1 - p_2 - p_3), \quad (\text{A1})$$

where $p_i = (p_i^0, \vec{p}_i) = (p_i^0, p_i^1, p_i^2, p_i^3)$. Furthermore, in the rest frame of top quark ($p_0^0 = m_t$), we have

$$\begin{aligned} \frac{d\Gamma}{ds_1 ds_2} &\propto \frac{1}{(2m_t)(2\pi)^5} d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1^2 - m_1^2) \delta(p_2^2 - m_2^2) \delta(p_3^2 - m_3^2) \theta(p_1^0) \theta(p_2^0) \cdot \\ &\quad \theta(p_3^0) \delta(s_1 - (p_1 + p_2)^2) \delta(s_2 - (p_2 + p_3)^2) \delta^4(p_0 - p_1 - p_2 - p_3) \\ &\propto \frac{1}{(2m_t)(2\pi)^5} d^4 p_1 d^4 p_3 \delta(p_1^2 - m_1^2) \delta((p_0 - p_1 - p_3)^2 - m_2^2) \delta(p_3^2 - m_3^2) \theta(p_1^0) \cdot \\ &\quad \theta(p_3^0) \theta(m_t - p_1^0 - p_3^0) \delta(s_1 - (p_0 - p_3)^2) \delta(s_2 - (p_0 - p_1)^2) \\ &\propto \frac{1}{2^8 m_t^3 \pi^5} d^3 \vec{p}_1 d^3 \vec{p}_3 \delta(p_1^{0^2} - \vec{p}_1^2 - m_1^2) \delta(p_3^{0^2} - \vec{p}_3^2 - m_3^2) \theta(p_1^0) \theta(p_3^0) \cdot \\ &\quad \theta(m_t - p_1^0 - p_3^0) \delta(s_2 + m_3^2 - m_2^2 - 2m_t p_3^0 + 2p_1^0 p_3^0 - 2\vec{p}_1 \cdot \vec{p}_3) \\ &\propto \frac{|\vec{p}_1| |\vec{p}_3|}{2^{10} m_t^3 \pi^5} d\Omega_1 \sin \theta_{13} d\theta_{13} d\phi_3 \theta(p_1^0) \theta(p_3^0) \theta(m_t - p_1^0 - p_3^0) \cdot \\ &\quad \delta(s_1 + s_2 - m_t^2 - m_2^2 + 2p_1^0 p_3^0 - 2|\vec{p}_1| \cdot |\vec{p}_3| \cos \theta_{13}) \end{aligned} \quad (\text{A2})$$

$$\propto \frac{1}{2^8 m_t^3 \pi^3} \theta(p_1^0) \theta(p_3^0) \theta(m_t - p_1^0 - p_3^0) \theta(X), \quad (\text{A3})$$

where $p_1^0 = \frac{m_t^2 + m_1^2 - s_2}{2m_t}$ and $p_3^0 = \frac{m_t^2 + m_3^2 - s_1}{2m_t}$, $|\vec{p}_3| = \sqrt{p_3^{0^2} - m_3^2}$ and $|\vec{p}_1| = \sqrt{p_1^{0^2} - m_1^2}$. The step function $\theta(X)$ is determined by ensuring $|\cos \theta_{13}| \leq 1$, where

$$\cos \theta_{13} = \frac{s_1 + s_2 - m_t^2 - m_2^2 + 2p_1^0 p_3^0}{2|\vec{p}_1| |\vec{p}_3|}.$$

All these step functions leads to the integration ranges:

$$s_1^{\min} = m_1^2 + m_2^2 - \frac{\left[(s_2 - m_t^2 + m_1^2)(s_2 - m_3^2 + m_2^2) + \sqrt{\lambda(s_2, m_t^2, m_1^2) \lambda(s_2, m_3^2, m_2^2)} \right]}{2s_2} \quad (\text{A4})$$

$$s_1^{\max} = m_1^2 + m_2^2 - \frac{\left[(s_2 - m_t^2 + m_1^2)(s_2 - m_3^2 + m_2^2) - \sqrt{\lambda(s_2, m_t^2, m_1^2) \lambda(s_2, m_3^2, m_2^2)} \right]}{2s_2} \quad (\text{A5})$$

$$s_2^{\min} = (m_2 + m_3)^2 \quad (\text{A6})$$

$$s_2^{\max} = (m_t - m_1)^2 \quad (\text{A7})$$

where $\lambda(x, y, z) = (x - y - z)^2 - 4yz$.

Furthermore, we can obtain the $\cos \theta_{13}$ distribution from Eq.(A2):

$$\frac{d\Gamma}{d\cos\theta_{13}ds_1} \propto \frac{J}{2^7 m_t^3 \pi^3} \theta(p_1^0) \theta(p_3^0) \theta(m_t - p_1^0 - p_3^0) \theta(Y), \quad (\text{A8})$$

where the extra Jacobian

$$J = \frac{-|\vec{p}_1||\vec{p}_3|}{\left| 1 - \frac{p_3^0}{2m_t} + \frac{|\vec{p}_3|(m_1^2 + m_t^2 - s_2) \cos \theta_{13}}{m_t \sqrt{m_1^4 + (m_t^2 - s_2)^2 - 2m_1^2(m_t^2 + s_2)}} \right|} \quad (\text{A9})$$

and

$$s_2 = \frac{1}{|\vec{p}_3|^2 \cos^2 \theta_{13} - (m_t - p_3^0)^2} \left\{ (m_1^2 + m_t^2) |\vec{p}_3|^2 \cos^2 \theta_{13} - (m_t - p_3^0)(m_t(m_2^2 - m_3^2 + m_t p_3^0) - m_1^2 p_3^0) - m_t |\vec{p}_3| \cos \theta_{13} \left[-2m_1^2(m_2^2 + 2(m_t - p_3^0)^2 - s_1 - 2|\vec{p}_3|^2 \cos^2 \theta_{13}) + m_1^4 + (m_2^2 - s_1)^2 \right]^{1/2} \right\} \quad (\text{A10})$$

The $\theta(X)$ function determines the boundary of s_1 :

$$s_{1min} \leq s_1 \leq (m_t - m_3)^2 \quad (\text{A11})$$

where

$$s_{1min} = \frac{m_2^2 m_t^2 + m_1^2 (m_t^2 \cos^2 \theta_{13} + m_3^2 (\cos^2 \theta_{13} - 1)) + m_1 m_t \sqrt{Y}}{m_t^2 - m_1^2 (1 - \cos^2 \theta_{13})} \quad (\text{A12})$$

with

$$Y = (m_1^2 - m_2^2 + m_3^2 - m_t^2)^2 - (m_1^4 - 2(m_2^2 - 3m_3^2 + m_t^2)m_1^2 + ((m_2 - m_3)^2 - m_t^2)((m_2 + m_3)^2 - m_t^2)) \cos^2 \theta_{13} + 4m_1^2 m_3^2 \cos^4 \theta_{13}. \quad (\text{A13})$$

It should be noted that the integration over $\cos \theta_{13}$ should be from 1 to -1 . The distribution for θ_{12} can be obtained in a similar way.

APPENDIX B: AMPLITUDE OF THE PROCESS $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$

Now let us illustrate the method to calculate the amplitude squared. In general, the amplitude for the process in which two massive fermions with spin projection s and s' are involved can be written as

$$M_{ss'} = \bar{u}_s(p_2) A u_{s'}(p_0), \quad (\text{B1})$$

where p_2 and p_0 denote the momenta of the top quark and the charm quark with mass m_t and m_c , and A is an explicit string of Dirac γ matrices for the process, which can be read from Eqs.(18,19,20,21,22, 23,24,25).

Let us introduce a massless spinor $u_-(k_0)$ with a light-like momentum k_0 and negative helicity first. Thus $u_-(k_0)$ is satisfied with the following projection relation:

$$u_-(k_0)\bar{u}_-(k_0) = \omega_- \not{k}_0, \quad (\text{B2})$$

where $\omega_- = (1 - \gamma_5)/2$. By introducing another spacelike vector k_1 which satisfies the relations:

$$k_1 \cdot k_1 = -1, k_0 \cdot k_1 = 0, \quad (\text{B3})$$

then the other, massless, independent and positive helicity spinor $u_+(k_0)$ may be constructed:

$$u_+(k_0) = \not{k}_1 u_-(k_0). \quad (\text{B4})$$

It is easy to check that $u_+(k_0)$ is satisfied with the projection

$$u_+(k_0)\bar{u}_+(k_0) = \omega_+ \not{k}_0, \quad (\text{B5})$$

where $\omega_+ = (1 + \gamma_5)/2$. Using these massless spinors, one can construct the massive spinors for the fermion and antifermion as follows:

$$u_s(q) = (\not{q} + m)u_-(k_0)/\sqrt{2k_0 \cdot q}, \quad (\text{B6})$$

$$u_{-s}(q) = (\not{q} + m)u_+(k_0)/\sqrt{2k_0 \cdot q}, \quad (\text{B7})$$

with the spin vector s_μ :

$$s_\mu = \frac{q_\mu}{m} - \frac{m}{q \cdot k_0} k_{0\mu}.$$

Using the above identities, we can write down the amplitude $M_{\pm s \pm s'}$ with four possible spin projections in the trace form for the γ -matrices:

$$M_{ss'} = NTr[(\not{p}_0 + m_t) \cdot \omega_- \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B8})$$

$$M_{-s-s'} = NTr[(\not{p}_0 + m_t) \cdot \omega_+ \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B9})$$

$$M_{-ss'} = NTr[(\not{p}_0 + m_t) \cdot \omega_- \not{k}_0 \not{k}_1 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B10})$$

$$M_{s-s'} = NTr[(\not{p}_0 + m_t) \cdot \omega_+ \not{k}_1 \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B11})$$

with the normalization constant $N = 1/\sqrt{4(k_0 \cdot p_0)(k_0 \cdot p_2)}$. Thus the squared unpolarized matrix elements can be written as

$$|M|^2 = |M_{ss'}|^2 + |M_{-s-s'}|^2 + |M_{-ss'}|^2 + |M_{s-s'}|^2.$$

Next, we are to simplify the calculation. For such purpose, we recombine these $M_{\pm s \pm s'}$ into M_n ($n = 1, \dots, 4$) as follows:

$$M_1 = \frac{1}{\sqrt{2}}(M_{ss'} + M_{-s-s'}) = \frac{N}{\sqrt{2}} \text{Tr}[(\not{p}_0 + m_t) \cdot \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B12})$$

$$M_2 = \frac{1}{\sqrt{2}}(M_{ss'} - M_{-s-s'}) = \frac{N}{\sqrt{2}} \text{Tr}[(\not{p}_0 + m_t) \cdot \not{k}_0 \gamma_5 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B13})$$

$$M_3 = \frac{1}{\sqrt{2}}(M_{s-s'} - M_{-ss'}) = \frac{N}{\sqrt{2}} \text{Tr}[(\not{p}_0 + m_t) \cdot \not{k}_1 \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B14})$$

$$M_4 = \frac{1}{\sqrt{2}}(M_{s-s'} + M_{-ss'}) = \frac{N}{\sqrt{2}} \text{Tr}[(\not{p}_0 + m_t) \cdot \gamma_5 \not{k}_1 \not{k}_0 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B15})$$

and then we have $|M|^2 = |M_1|^2 + |M_2|^2 + |M_3|^2 + |M_4|^2$. In order to write down A as explicitly and simply as possible, we set the vector k_0 :

$$k_0 = p_2 - \alpha p_0, \quad (\text{B16})$$

where the coefficient α is determined by the requirement that k_0 be a light-like vector:

$$\alpha = \frac{p_0 \cdot p_2}{m_t^2} \pm \frac{\Delta}{m_t^2}$$

with $\Delta = \sqrt{(p_0 \cdot p_2)^2 - m_t^2 m_c^2}$. Furthermore, if choosing k_1 : $k_1 \cdot p_0 = 0$ and $k_1 \cdot p_2 = 0$. And then the resultant M_n can be simplified as

$$M_1 = L_1 \text{Tr}[(\not{p}_0 + m_t) \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B17})$$

$$M_2 = -L_2 \text{Tr}[(\not{p}_0 + m_t) \cdot \gamma_5 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B18})$$

$$M_3 = L_2 \text{Tr}[(\not{p}_0 + m_t) \cdot \not{k}_1 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B19})$$

$$M_4 = L_1 \text{Tr}[(\not{p}_0 + m_t) \cdot \gamma_5 \cdot \not{k}_1 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B20})$$

with

$$2L_1 = \frac{1}{\sqrt{p_0 \cdot p_2 + m_c m_t}} \quad \text{and} \quad 2L_2 = \frac{1}{\sqrt{p_0 \cdot p_2 - m_c m_t}}.$$

The value of k_1 is arbitrary, and we take its explicit form as

$$k_1^\mu = i\kappa \epsilon^{\mu\nu\rho\sigma} p_{0\nu} p_{1\rho} p_{2\sigma}, \quad (\text{B21})$$

where κ is a suitable normalization constant and k_1 can be expressed as

$$k_1 = \kappa \gamma_5 [(p_0 \cdot p_1) \not{p}_2 + (p_2 \cdot p_1) \not{p}_0 - (p_2 \cdot p_0) \not{p}_1 - \not{p}_2 \cdot \not{p}_1 \cdot \not{p}_0].$$

Substituting k_1 into Eqs.(B19,B20), we obtain

$$M_3 = M'_3 + \kappa[(p_0 \cdot p_1)m_c - (p_1 \cdot p_2)m_t]M_2 \quad (\text{B22})$$

$$M_4 = M'_4 - \kappa[(p_0 \cdot p_1)m_c + (p_1 \cdot p_2)m_t]M_1 \quad (\text{B23})$$

where

$$M'_3 = \frac{\kappa}{4L_2} \text{Tr}[(\not{p}_0 + m_t) \cdot \gamma_5 \cdot \not{p}_1 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B24})$$

$$M'_4 = \frac{\kappa}{4L_1} \text{Tr}[(\not{p}_0 + m_t) \cdot \not{p}_1 \cdot (\not{p}_2 + m_c) \cdot A] \quad (\text{B25})$$

The amplitudes for the process $t(p_0) \rightarrow (b\bar{c})(p_1) + c(p_2) + W^+(p_3)$ can be expanded over some basic Lorentz structures:

$$M_i(n) = \sum_{j=1}^m A_j^i(n) B_j(n) (i = 1 - 4) \quad , \quad M'_i(n) = \sum_{j=1}^m A_j^{i'}(n) B_j(n) \quad (i = 3, 4) \quad (\text{B26})$$

where m is the number of basic Lorentz structure $B_j(n)$, whose value depends on the $(b\bar{c})$ -quarkonium state n : e.g. $m = 3$ for $n = (b\bar{c})[^1S_0]_1$, $m = 11$ for $n = (b\bar{c})[^3S_1]_1$ and $(b\bar{c})[^1P_1]_1$ and $m = 30$ for $n = (b\bar{c})[^3P_J]_1$. As for $A_j^3(n)$ and $A_j^4(n)$, they can be expressed by

$$A_j^3(n) = A_j^{3'}(n) + \kappa[(p_0 \cdot p_1)m_c - (p_1 \cdot p_2)m_t]A_j^2(n) \quad (\text{B27})$$

$$A_j^4(n) = A_j^{4'}(n) - \kappa[(p_0 \cdot p_1)m_c + (p_1 \cdot p_2)m_t]A_j^1(n) \quad (\text{B28})$$

The explicit expression for $A_j^{1,2}(n)$ and $A_j^{3',4'}(n)$ of each state shall be listed in the following subsections.

To short the notation, we define some dimensionless parameters

$$r_1 = \frac{m_b}{m_t}, \quad r_2 = \frac{m_c}{m_t}, \quad r_3 = \frac{m_w}{m_t}, \quad r_4 = \frac{M}{m_t}$$

and

$$\begin{aligned} u &= p_1 \cdot p_3 / m_t^2 = \frac{1}{2m_t^2}(s_3 - m_{B_c}^2 - m_w^2), \quad v = p_2 \cdot p_3 / m_t^2 = \frac{1}{2m_t^2}(s_2 - m_c^2 - m_w^2), \\ w &= p_0 \cdot p_3 / m_t^2 = \frac{1}{2m_t^2}(m_t^2 + m_w^2 - s_1), \quad x = p_1 \cdot p_2 / m_t^2 = \frac{1}{2m_t^2}(s_1 - m_{B_c}^2 - m_c^2), \\ y &= p_0 \cdot p_1 / m_t^2 = \frac{1}{2m_t^2}(m_t^2 + m_{B_c}^2 - s_2), \quad z = p_0 \cdot p_2 / m_t^2 = \frac{1}{2m_t^2}(m_t^2 + m_c^2 - s_3), \end{aligned}$$

where $s_1 = (p_1 + p_2)^2$, $s_2 = (p_2 + p_3)^2$ and $s_3 = (p_1 + p_3)^2$, which satisfy the relation: $s_1 + s_2 + s_3 = m_t^2 + m_c^2 + m_w^2 + m_{B_c}^2$. And the short notations for the denominators are

$$\begin{aligned} d_1 &= \frac{1}{(p_2 + p_{11}^0)^2} \frac{1}{(p_1 + p_2)^2 - m_b^2}, \quad d_2 = \frac{1}{(p_2 + p_{11}^0)^2} \frac{1}{(p_3 + p_{12}^0)^2 - m_t^2}, \\ d_3 &= \frac{m_t^2}{(p_2 + p_{11}^0)^4} \frac{1}{(p_1 + p_2)^2 - m_b^2}, \quad d_4 = \frac{m_t^2}{(p_2 + p_{11}^0)^4} \frac{1}{(p_3 + p_{12}^0)^2 - m_t^2}, \\ d_5 &= \frac{1}{(p_2 + p_{11}^0)^2} \frac{m_t^2}{((p_3 + p_{12}^0)^2 - m_t^2)^2}. \end{aligned}$$

Furthermore, the following relations are useful to short the expressions:

$$y + z + w = 1, \quad x + u + r_4^2 = y, \quad x + v + r_2^2 = z, \quad u + v + r_3^2 = w.$$

1. Coefficients for spin-singlet S-wave state: $(b\bar{c})[^1S_0]_1$

There are three basic Lorentz structures B_j for the case of $(b\bar{c})[^1S_0]_1$, which are

$$B_1 = \frac{p_1 \cdot \epsilon(p_3)}{m_t}, \quad B_2 = \frac{p_2 \cdot \epsilon(p_3)}{m_t}, \quad B_3 = \frac{i}{m_t^3} \varepsilon(p_1, p_2, p_3, \epsilon(p_3)), \quad (\text{B29})$$

where the short notation $\varepsilon(p_1, p_2, p_3, \epsilon(p_3)) = \varepsilon^{\mu\nu\rho\sigma} p_{1\mu} p_{2\nu} p_{3\rho} \epsilon_\sigma(p_3)$. The values of the coefficients A_j^1 and $A_j^{3'}$ are

$$\begin{aligned} A_1^1 &= \frac{2L_1 m_t^{\frac{7}{2}}}{r_4^{\frac{3}{2}}} \left(d_2(r_1^3 r_2 + r_1^2(2(v+x) + r_2 + 4r_2^2) - r_2(3z - 2v + r_2(u-w+2) - x) \right. \\ &\quad \left. + r_1(-v + r_2(-2 + 2u + 3v + 4x + r_2(-2 + 3r_2) + r_3^2))) + d_1 r_4(2x(r_1 + 2r_2) + \right. \\ &\quad \left. (v + r_2 - r_1 r_2 + 4r_2^2)r_4) \right) \end{aligned} \quad (\text{B30})$$

$$\begin{aligned} A_2^1 &= \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(-2x d_1(1 + r_2) + d_1(u + r_1 + r_1^2 + 3r_1 r_2 - r_2(3 + 2r_2))r_4 + \right. \\ &\quad \left. d_2(1 + r_2)(u + (1 + r_1)r_4) \right) \end{aligned} \quad (\text{B31})$$

$$A_3^1 = \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(d_2(1 + r_2) + d_1 r_4 \right) \quad (\text{B32})$$

$$\begin{aligned} A_1^{3'} &= \frac{\kappa m_t^{\frac{9}{2}}}{2L_2 r_4^{\frac{3}{2}}} \left(d_1 r_4(-2x(v + r_1 + r_2(2 + r_2)) + (v(r_1 - 3r_2) + r_2(r_1 + r_1^2 + 3r_1 r_2 - \right. \\ &\quad \left. r_2(3 + 2r_2)))r_4) + d_2(4x^2 r_1 + 2x(r_1^3 + r_2(-1 + v + r_2) + r_1^2(-1 + 3r_2) + \right. \\ &\quad \left. r_1(-1 + 2u + v + 2r_2^2)) + r_4((1 + r_1)r_2(2u + 2r_1^2 + (-1 + r_2)r_2 + r_1(-1 + 3r_2)) + \right. \\ &\quad \left. v(2u + (-1 + r_2)r_4))) \right) \end{aligned} \quad (\text{B33})$$

$$A_2^{3'} = \frac{-\kappa m_t^{\frac{9}{2}}}{2L_2 \sqrt{r_4}} \left((d_2(2u(u+x) + r_1^4 - (u+2x)r_2 - r_2^3 + r_1^3(-1 + 3r_2) + r_2^2(u - r_3^2) + \right.$$

$$r_1^2(2(u+x) + 3(-1+r_2)r_2 - r_3^2) + r_1(-u - 2x + r_2(3u + 2x + (-3+r_2)r_2 - 2r_3^2))) + d_1(-4x^2 + u(r_1 - 3r_2)r_4 + (-1+r_1 - 4r_2)r_4^3 - 2x(u + 4r_2r_4))) \quad (\text{B34})$$

$$A_3^{3'} = \frac{-\kappa m_t^{\frac{9}{2}}}{2L_2\sqrt{r_4}} \left((d_2(2(u+x) + 2r_1^2 + 3r_1r_2 + r_2^2 + r_4) + d_1(-2x + (r_1 - 3r_2)r_4)) \right) \quad (\text{B35})$$

And the values of the coefficients A_j^2 and $A_j^{4'}$ are

$$A_1^2 = -\frac{A_1^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{7}{2}}}{r_4^{\frac{3}{2}}} \left(2xd_2r_2 - d_2(-v + (r_1 - 3r_2)r_2)r_4 - d_1r_2r_4^2 \right) \quad (\text{B36})$$

$$A_2^2 = -\frac{A_2^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(d_1(-2x + r_1r_4 - 3r_2r_4) + d_2(u + r_4^2) \right) \quad (\text{B37})$$

$$A_3^2 = -\frac{A_3^1}{L_1}L_2 - \frac{4L_2d_2m_t^{\frac{7}{2}}}{\sqrt{r_4}} \quad (\text{B38})$$

$$A_1^{4'} = \frac{A_1^{3'}}{L_1}L_2 - \frac{\kappa m_t^{\frac{9}{2}}}{L_1\sqrt{r_4}} \left(d_2(r_1^2r_2 + r_2(2u - 2v + x + z) + r_1(z - 2v - 3x)) + d_1(r_1 - 3r_2)r_2r_4 - 2xd_1(r_2 + r_4) \right) \quad (\text{B39})$$

$$A_2^{4'} = \frac{A_2^{3'}}{L_1}L_2 - \frac{\kappa m_t^{\frac{9}{2}}}{L_1\sqrt{r_4}} \left(d_1r_4^3 + d_2r_4(x + y) \right) \quad (\text{B40})$$

$$A_3^{4'} = \frac{A_3^{3'}}{L_1}L_2 + \frac{\kappa m_t^{\frac{9}{2}}d_2\sqrt{r_4}}{L_1} \quad (\text{B41})$$

Here for convenience an overall factor \mathcal{C}_s has been contracted out from these coefficients. Then the square of the amplitude $|M_i|^2$ can be conveniently obtained with the help of Eqs.(B26,B27,B28).

2. Amplitude for spin-triplet S-wave state: $[^3S_1]_1$

There are eleven basic Lorentz structures B_j for the case of $(b\bar{c})[^3S_1]$, which are

$$\begin{aligned} B_1 &= \epsilon(s_z) \cdot \epsilon(p_3), \quad B_2 = \frac{i}{m_t^2} \epsilon(p_1, p_2, \epsilon(s_z), \epsilon(p_3)), \quad B_3 = \frac{i}{m_t^2} \epsilon(p_1, p_3, \epsilon(s_z), \epsilon(p_3)), \\ B_4 &= \frac{i}{m_t^2} \epsilon(p_2, p_3, \epsilon(s_z), \epsilon(p_3)), \quad B_5 = \frac{p_1 \cdot \epsilon(p_3)p_2 \cdot \epsilon(s_z)}{m_t^2}, \quad B_6 = \frac{p_1 \cdot \epsilon(p_3)p_3 \cdot \epsilon(s_z)}{m_t^2}, \\ B_7 &= \frac{p_2 \cdot \epsilon(p_3)p_2 \cdot \epsilon(s_z)}{m_t^2}, \quad B_8 = \frac{p_2 \cdot \epsilon(p_3)p_3 \cdot \epsilon(s_z)}{m_t^2}, \quad B_9 = \frac{ip_1 \cdot \epsilon(p_3)}{m_t^4} \epsilon(p_1, p_2, p_3, \epsilon(s_z)), \\ B_{10} &= \frac{i}{m_t^4} \epsilon(p_1, p_2, p_3, \epsilon(p_3))p_3 \cdot \epsilon(s_z), \quad B_{11} = \frac{i}{m_t^4} \epsilon(p_1, p_2, p_3, \epsilon(p_3))p_2 \cdot \epsilon(s_z), \end{aligned} \quad (\text{B42})$$

where the polarization vector $\epsilon(s_z)$ related to the spin angular momentum of the spin-triplet state. The values of the coefficients A_j^1 and $A_j^{3'}$ are

$$A_1^1 = \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(- (d_1 r_4 (x - r_1 (z + r_2) + r_2 (y + z + r_2))) + d_2 (2u(v + x) + x(r_1 - 1) + 2x(r_1 - 1)r_4 + v(2r_1 - 1)r_4 + 2r_2^2(u + (r_1 - 1)r_4) + r_2(u - x + (u + x)r_1 + (2r_1 + w - 2)r_4 + (-1 + r_1)r_4^2)) \right) \quad (B43)$$

$$A_2^1 = \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(d_2 (1 + 2u + r_1 + r_2 + r_1 r_2 + 2r_3^2) + d_1 (1 + r_1)r_4 \right) \quad (B44)$$

$$A_3^1 = \frac{2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(- d_1 r_2 r_4 + d_2 (4(v + x) + r_2 (1 - r_1 + 2r_2 + r_4)) \right) \quad (B45)$$

$$A_4^1 = \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(2ud_2 + d_1 (-r_1 + r_2)r_4 + d_2 r_4 (-1 - r_2 + 2r_4) \right) \quad (B46)$$

$$A_5^1 = \frac{-2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(d_2 (1 + r_1)(1 + r_2) + d_1 (1 + r_1 + 2r_2)r_4 \right) \quad (B47)$$

$$A_6^1 = \frac{2L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(- d_1 r_2 r_4 + d_2 (2(v + x) + r_2 (1 - r_1 + 2r_2 + r_4)) \right) \quad (B48)$$

$$A_7^1 = -4d_1 L_1 m_t^{\frac{7}{2}} \sqrt{r_4} (1 + r_2) \quad (B49)$$

$$A_8^1 = 2L_1 m_t^{\frac{7}{2}} \sqrt{r_4} (d_2 + d_1 r_1 - d_1 r_2 + d_2 r_2) \quad (B50)$$

$$A_9^1 = \frac{-4d_2 L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \quad (B51)$$

$$A_{10}^1 = \frac{4d_2 L_1 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \quad (B52)$$

$$A_1^{3'} = \frac{-\kappa m_t^{\frac{9}{2}}}{2L_2 \sqrt{r_4}} \left((d_1 r_4 (-x(2(u + x) + r_1)) - (u + x)r_2^2 + (v - x)r_4^2 + r_2(x + (u + x)r_1 + (1 + r_1)r_4^2)) + d_2 (2x(x + (u + x)r_1) + x(2(v - 1) - 2r_1 + r_3^2)r_4 - (v - x)(1 + r_1)r_4^2 + vr_4^3 - r_2^2 r_4(u + r_4 + r_1 r_4) + r_2(u + r_4 + r_1 r_4)(2(u + x) + r_4(-1 + 2r_4)))) \right) \quad (B53)$$

$$A_2^{3'} = \frac{\kappa m_t^{\frac{9}{2}}}{2L_2 \sqrt{r_4}} \left(- (d_1 r_4 (2x + r_2 + r_2^2 - r_1(1 + r_2) + r_4^2)) + d_2 (-2xr_1 + 2(u + x + ur_2) + r_4(r_3^2 + r_4 - r_1 r_4)) \right) \quad (B54)$$

$$A_3^{3'} = \frac{-\kappa m_t^{\frac{9}{2}}}{2L_2 \sqrt{r_4}} \left((d_1 (2x - r_1 r_2 + r_2^2)r_4 + d_2 (2x(1 + r_1) - 2ur_2 + 2vr_4 + r_2(1 + r_2 - 2r_4)r_4)) \right) \quad (B55)$$

$$A_4^{3'} = \frac{\kappa m_t^{\frac{9}{2}} \sqrt{r_4}}{2L_2} \left(d_1 r_4^2 + d_2 (2(u + x) + r_4(1 + r_1 + 2r_2 + r_4)) \right) \quad (B56)$$

$$A_5^{3'} = \frac{\kappa m_t^{\frac{9}{2}}}{2L_2\sqrt{r_4}} \left(d_1 r_4 (2v - 2x + r_1 + r_2 + r_1 r_2 + r_2^2 - r_4^2) + d_2 (2(u + x) - 2x r_1 + r_4 (r_3^2 + r_4 - r_1 r_4)) \right) \quad (\text{B57})$$

$$A_6^{3'} = \frac{-\kappa m_t^{\frac{9}{2}}}{2L_2\sqrt{r_4}} \left((d_1 (2x - r_1 r_2 + r_2^2) r_4 + d_2 (r_2^2 r_4 + 2(x + x r_1 + v r_4) + r_2 (-2(u + x) + r_4 - 2r_4^2))) \right) \quad (\text{B58})$$

$$A_7^{3'} = -\frac{\kappa d_1 m_t^{\frac{9}{2}} \sqrt{r_4}}{L_2} (u + 2x + r_4^2) \quad (\text{B59})$$

$$A_8^{3'} = \frac{\kappa m_t^{\frac{9}{2}} \sqrt{r_4}}{2L_2} \left(d_1 r_4^2 + d_2 (2(u + x) + r_4 (1 + r_1 + r_4)) \right) \quad (\text{B60})$$

$$A_9^{3'} = \frac{\kappa d_2 m_t^{\frac{9}{2}} r_2}{L_2 \sqrt{r_4}} \quad (\text{B61})$$

$$A_{11}^{3'} = -\frac{\kappa d_1 m_t^{\frac{9}{2}} \sqrt{r_4}}{L_2} \quad (\text{B62})$$

and $A_{11}^1 = A_{10}^{3'} = 0$. And the values of the coefficients A_j^2 and $A_j^{4'}$ are

$$A_1^2 = -\frac{A_1^1}{L_1} L_2 + \frac{4L_2 m_t^{\frac{7}{2}}}{\sqrt{r_4}} \left(d_1 (x - r_1 r_2 + r_2^2) r_4 + d_2 ((v + 2x) r_4 + 2r_2^2 r_4 - r_1 (x + 2r_2 r_4) + r_2 (y - 2u)) \right) \quad (\text{B63})$$

$$A_2^2 = -\frac{A_2^1}{L_1} L_2 - \frac{4L_2 m_t^{\frac{7}{2}}}{\sqrt{r_4}} (d_1 r_4 + d_2 r_4) \quad (\text{B64})$$

$$A_3^2 = -\frac{A_3^1}{L_1} L_2 + \frac{4d_2 L_2 m_t^{\frac{7}{2}} r_2}{\sqrt{r_4}} \quad (\text{B65})$$

$$A_4^2 = -\frac{A_4^1}{L_1} L_2 + 4d_2 L_2 m_t^{\frac{7}{2}} \sqrt{r_4} \quad (\text{B66})$$

$$A_5^2 = -\frac{A_5^1}{L_1} L_2 - \frac{4L_2 m_t^{\frac{7}{2}}}{\sqrt{r_4}} (d_1 r_4 + d_2 r_4) \quad (\text{B67})$$

$$A_6^2 = -\frac{A_6^1}{L_1} L_2 + \frac{4d_2 L_2 m_t^{\frac{7}{2}} r_2}{\sqrt{r_4}} \quad (\text{B68})$$

$$A_7^2 = -\frac{A_7^1}{L_1} L_2 - 8d_1 L_2 m_t^{\frac{7}{2}} \sqrt{r_4} \quad (\text{B69})$$

$$A_8^2 = -\frac{A_8^1}{L_1} L_2 + 4d_2 L_2 m_t^{\frac{7}{2}} \sqrt{r_4} \quad (\text{B70})$$

$$A_1^{4'} = \frac{A_1^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{9}{2}}}{L_1 \sqrt{r_4}} \left(-2x^2 d_2 - x r_4 (-(d_1 + 2d_2)(r_1 - r_2) + d_2 r_4) + r_4 (-u d_2 r_2 + v d_2 r_4 + r_2 r_4 (-d_1 r_4 - d_2 r_4)) \right) \quad (\text{B71})$$

$$A_2^{4'} = \frac{A_2^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{9}{2}}}{L_1 \sqrt{r_4}} (d_1 (r_1 - r_2) r_4 + d_2 (2(u + x) + r_4^2)) \quad (\text{B72})$$

$$A_3^{4'} = \frac{A_3^{3'}}{L_1} L_2 + \frac{\kappa d_2 m_t^{\frac{9}{2}}}{L_1 \sqrt{r_4}} (2x + r_2 r_4) \quad (\text{B73})$$

$$A_4^{4'} = \frac{A_4^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{9}{2}} r_4^{\frac{3}{2}}}{L_1} \quad (\text{B74})$$

$$A_5^{4'} = \frac{A_5^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{9}{2}}}{L_1 \sqrt{r_4}} (d_1 r_4^2 + d_2 (u + x + y)) \quad (\text{B75})$$

$$A_6^{4'} = \frac{A_6^{3'}}{L_1} L_2 + \frac{\kappa d_2 m_t^{\frac{9}{2}}}{L_1 \sqrt{r_4}} (2x + r_2 r_4) \quad (\text{B76})$$

$$A_8^{4'} = \frac{A_8^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{9}{2}} r_4^{\frac{3}{2}}}{L_1} \quad (\text{B77})$$

and for $j = 9, 10, 11$ and $m = 7, 9, 10, 11$, we have

$$A_j^2 = -\frac{A_j^1}{L_1} L_2 \quad \text{and} \quad A_m^{4'} = \frac{A_m^{3'}}{L_1} L_2.$$

Here for convenience an overall factor \mathcal{C}_s has been contracted out from these coefficients. Then the square of the amplitude $|M_i|^2$ can be conveniently obtained with the help of Eqs.(B26,B27,B28) and Eq.(33).

3. Amplitude for spin-singlet P-wave state: $[^1P_1]_1$

The basic structures are similar to the case of 3S_1 , only one needs to replace the polarization vector $\epsilon(s_z)$ related to the spin angular momentum of the spin-triplet S-state to the present $\epsilon(l_z)$ that is related to the radial angular momentum of the spin-singlet P-wave state. Secondly, we need to change coefficients there to the present case. The values of the coefficients A_j^1 and $A_j^{3'}$ are

$$\begin{aligned} A_1^1 = & \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} \left(- (d_1 (r_1 - r_2) r_4^2 (x + r_1 (r_2 + z) + r_2 (r_2 + y + z))) - 2d_5 r_1 r_2 (2ur_1 + \right. \\ & (r_1^2 + r_3^2 - 1)r_4)(x + 2(v + x)r_4 + 2r_2^2 r_4 + r_2(2r_4 + y)) + d_2 r_4 (r_1^2 (x + yr_2) \\ & + r_4(x - 2u(v + x) + 2r_2^3 - r_2^2(-2 + 3u + v + r_3^2) + r_2(v - u + 3x + r_4^2)) \\ & - r_1(r_2(x - (w - 2)r_4 + 2r_4^2) + r_4(v + 2vr_4 + 2x(1 + r_4)) \\ & \left. + r_2^2(u + x + r_4(2 + 3r_4)))) \right) \quad (\text{B78}) \end{aligned}$$

$$\begin{aligned} A_2^1 = & \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (-d_1 (r_1 - 1)(r_1 - r_2) r_4^2 + 2d_5 r_1 r_2 (1 + r_2)(2ur_1 + (r_1^2 + r_3^2 - 1)r_4) - \\ & d_2 r_4 (-r_1^2(1 + r_2) + r_1 r_2(1 + r_2) + r_4 + (r_2 + 2(u + r_3^2))r_4)) \quad (\text{B79}) \end{aligned}$$

$$A_3^1 = \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (d_2 r_4 (r_1^2 r_2 + (r_2 + v + x + 3z) r_4 - r_1 r_2 (r_2 + r_4)) + r_2 (d_1 (r_1 - r_2) r_4^2 + 2d_5 r_1 r_2 (2ur_1 + (r_1^2 + r_3^2 - 1) r_4))) \quad (B80)$$

$$A_4^1 = -\frac{L_1 m_t^{\frac{5}{2}} \sqrt{r_4}}{r_1 r_2} (d_1 (r_2^2 - r_1^2) + d_2 (r_1 (1 + r_2) - r_2 (1 + r_2) + 2(u + r_4^2))) \quad (B81)$$

$$A_5^1 = -\frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (d_2 (1 + r_2) (r_2 + r_1 (1 - r_1 + r_2)) r_4 + d_1 (-1 + r_1 - 2r_2) (r_1 - r_2) r_4^2 + 2r_1 r_2 (d_5 (1 + r_2) (2ur_1 + (r_1^2 + r_3^2 - 1) r_4) + 2(d_4 (r_1^3 r_2 + r_1^2 (r_2 - 2(v + x - 2z)) - r_2 (r_2 (u - w + 2) - 2v - x + 3z) + r_1 (-v + r_2 (-2 + 2u + 3v + 4x + r_2 (-2 + 3r_2) + r_3^2))) + d_3 r_4 (2x(r_1 + 2r_2) + (v + r_2 - r_1 r_2 + 4r_2^2) r_4)))) \quad (B82)$$

$$A_6^1 = \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (d_2 r_4 (r_1^2 r_2 + (r_2 - v - x + 3z) r_4 - r_1 r_2 (r_2 + r_4)) + r_2 (d_1 (r_1 - r_2) r_4^2 + 2d_5 r_1 (2v(2r_1 - 1) r_4 + 4(r_1 - 1) r_2^2 r_4 + 4x(r_1 + (r_1 - 1) r_4) + r_2 (2(u + 2x) r_1 + (-3 + 2v - (-4 + r_1) r_1 + r_3^2) r_4 + 2(-1 + r_1) r_4^2)))) \quad (B83)$$

$$A_7^1 = \frac{2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_1 (r_1 - r_2) (1 + r_2) r_4 + 2r_1 r_2 (d_4 (1 + r_2) (u + r_4 + r_1 r_4) - d_3 (2x - 2ur_4 + 2r_2^2 r_4 + r_4^2 - 2r_4^3 + r_1 (u - 2(1 + r_2) r_4 + r_4^2) + r_2 (2r_4 - u + 2y)))) \quad (B84)$$

$$A_8^1 = -\frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_2 (r_1 - r_2) (1 + r_2 - r_4) r_4 + 4d_5 r_1 r_2 (1 + r_2) (u + r_4 + r_1 r_4)) \quad (B85)$$

$$A_9^1 = \frac{-2d_2 L_1 m_t^{\frac{5}{2}} \sqrt{r_4}}{r_1 r_2} \quad (B86)$$

$$A_{10}^1 = \frac{2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (-2d_5 r_1 r_2 (1 + r_2) + d_2 r_4) \quad (B87)$$

$$A_{11}^1 = \frac{4L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (d_4 (1 + r_2) + d_3 r_4) \quad (B88)$$

$$A_1^{3'} = \frac{-\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} (d_1 (r_1 - r_2) r_4^2 (2x(u + x) - x r_1 + (u + x) r_2^2 + (x - v) r_4^2 + r_2 (x(r_1 - 1) + u r_1 + (r_1 - 1) r_4^2)) - 2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2) r_4) (-2x^2 + r_4 (-2ur_2 + v r_4 + r_2 (1 + r_2 - 2r_4) r_4) - x(2u + r_4 (-2 + 2r_2 + r_4))) + d_2 r_4 (2x^2 (r_1 (-r_1 + r_2) + r_4) + x(-r_1^2 (2u + r_4^2) + r_4 (r_2 (2(-1 + u + v) + 2r_2 + r_3^2) + r_4^2) + r_1 (2ur_2 - (-2 + 2v + 2r_2 + r_3^2) r_4 + (-2 + 3r_2) r_4^2)) + r_4 (-v r_4 (- (r_1 - r_2) (r_1 - r_4) + r_4) + r_2 (u(2u + r_2 - r_2^2) + r_1^2 (1 + r_2) r_4 + (-1 + 2u + r_2) r_4^2 + u r_1 (-1 + r_2 + 2r_4) - r_1 r_4 (r_2 + r_2^2 - 2(-1 + r_4) r_4)))))) \quad (B89)$$

$$A_2^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} (2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2) r_4) (2x + r_4^2) + d_1 (r_1 - r_2) r_4^2 (2x + r_1 +$$

$$r_2 + r_1 r_2 + r_2^2 + r_4^2) + d_2 r_4 (2x(r_1^2 - r_1 r_2 + r_4) + r_4 (2u(1 + r_2) + r_2 r_3^2 + r_1^2 r_4 + r_4^2 - r_1(r_3^2 + r_2 r_4)))) \quad (\text{B90})$$

$$A_3^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} \left(d_1 (r_1 - r_2) (2x + r_2 (r_1 + r_2)) r_4^2 + 4x d_5 r_1 r_2 (2ur_1 + (r_1^2 + r_3^2 - 1)r_4) + d_2 r_4 (2v(r_1 - r_2)r_4 - 2x(r_1(r_2 - r_1) + r_4) + r_2 r_4 (r_1(1 + r_2) - r_2(1 + r_2) + 2(u + r_4^2))) \right) \quad (\text{B91})$$

$$A_4^{3'} = \frac{-\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{4L_2 r_1 r_2} \left(d_1 (r_1 - r_2) r_4^2 + 2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + d_2 (-2(u + x)r_2 + r_1^2 r_4 - (1 + 3r_2)r_4^2 + r_1(2(u + x) - r_2 r_4 + r_4^2)) \right) \quad (\text{B92})$$

$$A_5^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} \left(d_1 (r_1 - r_2) (2x - 2v + r_1 + r_1^2 - r_2 + 3r_1 r_2) r_4^2 + d_2 r_4 ((2u + r_1^2 + r_1^3 + r_2(r_2 + r_3^2) - r_1((-2 + r_2)r_2 + r_3^2))r_4 + 2x(r_1^2 - r_1 r_2 + r_4)) - 2r_1 r_2 (d_5 (2ur_1 + (-1 + r_1^2 + r_3^2)r_4)(u + x + y) + 2(d_3 r_4 (-2x(v + r_1 + r_2(2 + r_2)) + (v(r_1 - 3r_2) + r_2(r_1 + r_1^2 + 3r_1 r_2 - r_2(3 + 2r_2))))r_4) + d_4 (4x^2 r_1 + 2x(r_1^3 + r_2(-1 + v + r_2) + r_1^2(3r_2 - 1) + r_1(2u - 1 + v + 2r_2^2)) + r_4((1 + r_1)r_2(2u + 2r_1^2 + (r_2 - 1)r_2 + r_1(-1 + 3r_2)) + v(2u + (-1 + r_2)r_4)))) \right) \quad (\text{B93})$$

$$A_6^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} \left(d_1 (r_1 - r_2) (2x + r_2 (r_1 + r_2)) r_4^2 + 4d_5 r_1 r_2 (4x^2 r_1 - (1 + r_1)r_2 r_4 (-2u + (1 + r_2 - 2r_4)r_4) + v r_4 (2u - (1 + r_1)r_4 + r_4^2) + 2x(2ur_1 + (v + (1 + r_1)(r_2 - 1))r_4 + r_1 r_4^2)) + d_2 r_4 (2v(r_1 - r_2)r_4 + 2x(r_1^2 - r_1 r_2 + (-1 + r_2)r_4) + r_2 r_4 (r_1(1 + r_2) - r_2(1 + r_2) + 2(u + r_4^2))) \right) \quad (\text{B94})$$

$$A_7^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 \sqrt{r_4}} \left(d_1 (r_1 - r_2) r_4 (x + y) + 2d_4 r_1 r_2 (2u(u + x) + (u + 2x)(r_1 - 1)r_4 + u r_4^2 - r_3^2 r_4^2 + (r_1 - 1)r_4^3) + 2d_3 r_1 r_2 (-4x^2 + 2x(-u + 2r_1 r_4 - 2r_4(r_2 + r_4)) + r_4(r_4(r_2 + r_2^2 - 4r_2 r_4 - r_4(2 + r_4) + r_1(1 + r_2 + 2r_4)) - u(2r_2 + r_4 - 2r_1))) \right) \quad (\text{B95})$$

$$A_8^{3'} = \frac{-\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 \sqrt{r_4}} \left(d_1 (r_1 - r_2) r_4^3 + 2d_5 r_1 r_2 (4u(u + x) + (4x(r_1 - 1) + u(4r_1 - 2))r_4 + (2u - 1 + r_1^2 - r_3^2)r_4^2 + 2(-1 + r_1)r_4^3) + d_2 r_4 (-2(u + x)r_2 + r_1^2 r_4 - (1 + r_2)r_4^2 + r_1(2(u + x) - r_2 r_4 + r_4^2)) \right) \quad (\text{B96})$$

$$A_9^{3'} = \frac{-\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_4^{\frac{3}{2}}} \left(-d_2 r_4^2 + 2d_5 r_1 (2ur_1 + (-1 + r_1^2 + r_3^2)r_4) \right) \quad (\text{B97})$$

$$A_{10}^{3'} = -\frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (2(u + x) + r_4(1 + r_1 + r_4)) \quad (\text{B98})$$

$$A_{11}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 \sqrt{r_4}} \left(d_1(r_1 - r_2)r_4 + 2r_1 r_2(-2xd_3 + d_3(2r_1 - 2r_2 - r_4)r_4 + d_4(2(u + x) + r_4(1 + r_1 + r_4))) \right) \quad (\text{B99})$$

And the values of the coefficients A_j^2 and $A_j^{4'}$ are

$$A_1^2 = -\frac{A_1^1}{L_1} L_2 + \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (-d_1(r_1 - r_2)r_4^2(x + r_2 r_4) - 2d_5 r_1 r_2(x + 2r_2 r_4)(2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + d_2 r_4(xr_1^2 + r_2 r_4(y - 2u - v + 2z) - r_1(x(r_2 + 2r_4) + r_4(v + 2r_2(r_2 + r_4)))))) \quad (\text{B100})$$

$$A_2^2 = -\frac{A_2^1}{L_1} L_2 + \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (4ud_5 r_1^2 r_2 + r_4(r_1(d_2(r_1 - r_2) + 2d_5 r_2(-1 + r_1^2 + r_3^2)) + (d_1 r_1 - (d_1 + d_2)r_2)r_4)) \quad (\text{B101})$$

$$A_3^2 = -\frac{A_3^1}{L_1} L_2 + \frac{2d_2 L_2 m_t^{\frac{5}{2}} \sqrt{r_4}}{r_1} \quad (\text{B102})$$

$$A_4^2 = -\frac{A_4^1}{L_1} L_2 + \frac{2d_2 L_2 m_t^{\frac{5}{2}} \sqrt{r_4}}{r_1 r_2} (r_2 - r_1) \quad (\text{B103})$$

$$A_5^2 = -\frac{A_5^1}{L_1} L_2 - \frac{L_2 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (8ud_5 r_1^2 r_2 + 8d_4 r_1 r_2(-2xr_2 + (-v + (r_1 - 3r_2)r_2)r_4) + r_4(4d_5 r_1 r_2(-1 + r_1^2 + r_3^2) + 2(-(d_1 r_1) + d_1 r_2 + 4d_3 r_1 r_2^2)r_4 + d_2(-2r_1^2 + 3r_1 r_2 + r_2^2 + r_4 + (-1 + r_2)r_4))) \quad (\text{B104})$$

$$A_6^2 = -\frac{A_6^1}{L_1} L_2 + \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_4^{\frac{3}{2}}} (d_2 r_4^2 + 4d_5 r_1(2r_1 x + r_2 r_4) - r_4(r_2 r_4 - v + 2z)) \quad (\text{B105})$$

$$A_7^2 = -\frac{A_7^1}{L_1} L_2 + \frac{4L_2 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_1(r_1 - r_2)r_4 + 2r_1 r_2(-2xd_3 + d_3(2r_1 - 2r_2 - r_4)r_4 + d_4(u + r_4^2))) \quad (\text{B106})$$

$$A_8^2 = -\frac{A_8^1}{L_1} L_2 - \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_2(r_1 - r_2)r_4 + 4d_5 r_1 r_2(u + r_4^2)) \quad (\text{B107})$$

$$A_{10}^2 = -\frac{A_{10}^1}{L_1} L_2 - \frac{8d_5 L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} \quad (\text{B108})$$

$$A_{11}^2 = -\frac{A_{11}^1}{L_1} L_2 + \frac{8d_4 L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} \quad (\text{B109})$$

$$A_1^{4'} = \frac{A_1^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{2L_1 r_1 r_2 \sqrt{r_4}} (2d_5 r_1 r_2(2x + r_2 r_4)(2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + d_1(r_1 - r_2)r_4(xr_1 + r_2(y - u)) + d_2 r_4(-2x^2 - r_1^2 r_2 r_4 + vr_4^2 - xr_4^2 + 2xr_1(r_2 + r_4) - r_2^2(y + x) + r_1 r_2(u + r_4(r_2 + 2r_4)))) \quad (\text{B110})$$

$$A_2^{4'} = \frac{A_2^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (d_1(r_1 - r_2)r_4 + d_2(y + u + x)) \quad (\text{B111})$$

$$A_3^{4'} = \frac{A_3^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (-2x + (r_1 - r_2)r_2) \quad (\text{B112})$$

$$A_4^{4'} = \frac{A_4^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{7}{2}} r_4^{\frac{5}{2}}}{2L_1 r_1 r_2} \quad (\text{B113})$$

$$A_5^{4'} = \frac{A_5^{3'}}{L_1} L_2 + \frac{\kappa m_t^{\frac{7}{2}}}{2L_1 r_1 r_2 r_4^{\frac{3}{2}}} (2d_4 r_2 (-4x r_1 (r_1 - r_2) r_4 + r_1 r_4 (-2v r_4 + r_2 (4u + 3r_1^2 + r_2 (2r_2 - 1) + r_1 (5r_2 - 1 - r_4) + r_4))) + r_4 (-x (8d_3 r_1 r_2 (r_1 + 2r_2) + 2d_2 r_4) - r_4 (d_1 (r_1 - r_2)^2 - 4d_3 r_1 (r_1 - 3r_2) r_2^2 + d_2 (2u + r_4^2)))) \quad (\text{B114})$$

$$A_6^{4'} = \frac{A_6^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{2L_1 r_1 r_2 \sqrt{r_4}} (d_2 (-2x + (r_1 - r_2)r_2) r_4 - 4d_5 r_1 r_2 (v r_4 + r_2^2 r_4 + r_1 (2x + r_2 r_4) - 2y r_2)) \quad (\text{B115})$$

$$A_7^{4'} = \frac{A_7^{3'}}{L_1} L_2 + \frac{2\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{L_1} (d_3 r_4^2 + d_4 (x + y)) \quad (\text{B116})$$

$$A_8^{4'} = \frac{A_8^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (d_2 r_4^2 + 4d_5 r_1 r_2 (x + y)) \quad (\text{B117})$$

$$A_{10}^{4'} = \frac{A_{10}^{3'}}{L_1} L_2 + \frac{2\kappa d_5 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_1} \quad (\text{B118})$$

$$A_{11}^{4'} = \frac{A_{11}^{3'}}{L_1} L_2 - \frac{2\kappa d_4 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_1} \quad (\text{B119})$$

and

$$A_{10}^2 = -\frac{A_{10}^1}{L_1} L_2 \quad \text{and} \quad A_{10}^{4'} = \frac{A_{10}^{3'}}{L_1} L_2. \quad (\text{B120})$$

Here for convenience an overall factor \mathcal{C}_s has been contracted out from these coefficients. Then the square of the amplitude $|M_i|^2$ can be conveniently obtained with the help of Eqs.(B26,B27,B28) and Eq.(33).

4. Amplitude for spin-triplet P-wave state: $[^3P_J]_1$

There are totally thirty independent basic Lorentz structures B_j for the case of $(b\bar{c})[(^3P_J)_1]$, which are

$$\begin{aligned} B_1 &= \frac{1}{m_t} p_{2\alpha} \epsilon_\beta(p_3) \varepsilon_{\alpha\beta}^J, \quad B_2 = \frac{1}{m_t} p_2 \cdot \epsilon(p_3) \varepsilon_{\alpha\alpha}^J, \quad B_3 = \frac{1}{m_t} p_1 \cdot \epsilon(p_3) \varepsilon_{\alpha\alpha}^J, \\ B_4 &= \frac{1}{m_t} p_{3\alpha} \epsilon_\beta(p_3) \varepsilon_{\alpha\beta}^J, \quad B_5 = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_2, p_3, \alpha) \epsilon_\beta(p_3), \quad B_6 = \frac{i\varepsilon_{\alpha\alpha}^J}{m_t^3} \varepsilon(p_1, p_2, p_3, \epsilon(p_3)), \\ B_7 &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_2, \alpha, \epsilon(p_3)) p_{3\beta}, \quad B_8 = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_2, \alpha, \epsilon(p_3)) p_{2\beta}, \\ B_9 &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_2, \alpha, \beta) p_1 \cdot \epsilon(p_3), \quad B_{10} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_3, \alpha, \epsilon(p_3)) p_{2\beta}, \end{aligned}$$

$$\begin{aligned}
B_{11} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_3, \alpha, \epsilon(p_3)) p_{3\beta}, \quad B_{12} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_3, \alpha, \beta) p_1 \cdot \epsilon(p_3), \\
B_{13} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, p_3, \alpha, \beta) p_2 \cdot \epsilon(p_3), \quad B_{14} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_2, p_3, \alpha, \epsilon(p_3)) p_{2\beta}, \\
B_{15} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_2, p_3, \alpha, \epsilon(p_3)) p_{3\beta}, \quad B_{16} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_2, p_3, \alpha, \beta) p_1 \cdot \epsilon(p_3), \\
B_{17} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_1, \alpha, \epsilon(p_3), \beta), \quad B_{18} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_2, \alpha, \epsilon(p_3), \beta), \quad B_{19} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^3} \varepsilon(p_3, \alpha, \epsilon(p_3), \beta) \\
B_{20} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^5} \varepsilon(p_1, p_2, p_3, \alpha) p_1 \cdot \epsilon(p_3) p_{2\beta}, \quad B_{21} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^5} \varepsilon(p_1, p_2, p_3, \alpha) p_1 \cdot \epsilon(p_3) p_{3\beta}, \\
B_{22} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^5} \varepsilon(p_1, p_2, p_3, \cdot \epsilon(p_3)) p_{3\alpha} p_{2\beta}, \quad B_{23} = \frac{i\varepsilon_{\alpha\beta}^J}{m_t^5} \varepsilon(p_1, p_2, p_3, \cdot \epsilon(p_3)) p_{3\alpha} p_{3\beta}, \\
B_{24} &= \frac{i\varepsilon_{\alpha\beta}^J}{m_t^5} \varepsilon(p_1, p_2, p_3, \cdot \epsilon(p_3)) p_{2\alpha} p_{2\beta}, \quad B_{25} = \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_1 \cdot \epsilon(p_3) p_{2\alpha} p_{3\beta}, \\
B_{26} &= \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_1 \cdot \epsilon(p_3) p_{3\alpha} p_{3\beta}, \quad B_{27} = \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_1 \cdot \epsilon(p_3) p_{2\alpha} p_{2\beta}, \quad B_{28} = \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_2 \cdot \epsilon(p_3) p_{2\alpha} p_{2\beta}, \\
B_{29} &= \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_2 \cdot \epsilon(p_3) p_{3\alpha} p_{2\beta}, \quad B_{30} = \frac{\varepsilon_{\alpha\beta}^J}{m_t^3} p_2 \cdot \epsilon(p_3) p_{3\alpha} p_{3\beta}. \tag{B121}
\end{aligned}$$

Noting the fact that $\varepsilon_{\alpha\beta}^{0,2}$ is the symmetric tensor and $\varepsilon_{\alpha\beta}^1$ is the anti-symmetric tensor, and the fact that $\varepsilon_{\alpha\alpha}^1 = \varepsilon_{\alpha\alpha}^2 = 0$, one may observe that the terms involving the following coefficients do not have any contributions to the square of the amplitude, so one can safely set the following coefficients to zero:

$$A_j^i(^3P_0) = 0 \quad \text{for } i = (1-4), j = (9, 12, 13, 16, 17, 18, 19) \tag{B122}$$

$$A_j^i(^3P_1) = 0 \quad \text{for } i = (1-4), j = (2, 3, 6, 24, 26, 27, 28, 30) \tag{B123}$$

$$A_j^i(^3P_2) = 0 \quad \text{for } i = (1-4), j = (2, 3, 6, 9, 12, 13, 16, 17, 18, 19). \tag{B124}$$

The coefficients A_j^1 , A_j^2 , $A_j^{3'}$ and $A_j^{4'}$ that are the same for all the three $[^3P_J]_1$ states (with $J = 1, 2, 3$):

$$A_{20}^1 = \frac{8d_4 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}}, \quad A_{21}^1 = \frac{-8d_5 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}}, \quad A_{22}^1 = \frac{-8d_4 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}} \tag{B125}$$

$$A_{23}^1 = \frac{8d_5 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}}, \quad A_{20}^{3'} = \frac{-2\kappa d_4 m_t^{\frac{7}{2}} r_2}{L_2 \sqrt{r_4}}, \quad A_{21}^{3'} = \frac{2\kappa d_5 m_t^{\frac{7}{2}} r_2}{L_2 \sqrt{r_4}} \tag{B126}$$

$$A_{22}^{3'} = A_{23}^{3'} = 0 \tag{B127}$$

and for $j = 20, 21, 22, 23$: $A_j^2 = -\frac{A_j^1}{L_1} L_2$ and $A_j^{4'} = \frac{A_j^{3'}}{L_1} L_2$.

The coefficients A_j^1 , A_j^2 , $A_j^{3'}$ and $A_j^{4'}$ that are the same for both $[^3P_0]_1$ and $[^3P_0]_2$:

$$A_1^1 = \frac{4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(d_5(1+r_2)(2ur_1+(r_1^2+r_3^2-1)r_4)-d_3r_4(x-r_1(z+r_2)+r_2(r_2+y+z))+d_4(2u(v+x)+x(r_1-1)+2x(r_1-1)r_4+v(2r_1-1)r_4+2r_2^2(u+(r_1-1)r_4)+r_2(u-x+(u+x)r_1+(w-2+2r_1)r_4+(r_1-1)r_4^2))) \quad (\text{B128})$$

$$A_4^1 = \frac{-4d_5L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(-x+2u(v+x)+(u-v-3x)r_2+(-1+3u+v)r_2^2-3r_2^3+r_1^2(r_2-v-x+3z)+r_1(-v-x+r_2(-1+3v+3x+r_2(-2+3r_2)))) \quad (\text{B129})$$

$$A_5^1 = 0 \quad (\text{B130})$$

$$A_7^1 = \frac{-4d_5L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(1+2u+r_1r_2+2r_3^2+r_4) \quad (\text{B131})$$

$$A_8^1 = \frac{4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(d_3(1+r_1)r_4+d_4(1+2u+r_1r_2+2r_3^2+r_4)) \quad (\text{B132})$$

$$A_{10}^1 = \frac{-4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(d_4(r_2+v+x+3z)-d_3r_2r_4) \quad (\text{B133})$$

$$A_{11}^1 = \frac{4d_5L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(r_2+v+x+3z) \quad (\text{B134})$$

$$A_{14}^1 = \frac{4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(2ud_4+(d_3(-r_1+r_2)+d_4(-1+2r_1+r_2))r_4) \quad (\text{B135})$$

$$A_{15}^1 = \frac{-4d_5L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(2u+2r_1^2+(-1+r_2)r_2+r_1(-1+3r_2)) \quad (\text{B136})$$

$$A_{25}^1 = \frac{-4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(d_5(1+r_1)(1+r_2)+d_4(r_2-v-x+3z)-d_3r_2r_4) \quad (\text{B137})$$

$$A_{29}^1 = -4L_1m_t^{\frac{5}{2}}\sqrt{r_4}(d_4+d_3r_1-d_3r_2+d_4r_2) \quad (\text{B138})$$

$$A_1^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{L_2\sqrt{r_4}}(d_4(2x^2+2u^2r_2+2r_1^4r_2+(u-v+3x)r_2^2+(-1+u+v)r_2^3+r_2^4+r_1^3(x+r_2+5r_2^2)+r_1^2(-v-x+r_2(3r_2+4u-3v+4z-1))+xr_2(u+v+w-2)+r_1((u-2v+2x)r_2+(-2+5u+2v+3x)r_2^2+3r_2^3+r_2^4+x(u+v+w+2x-2))))+d_3(-2x(u+x)+xr_2+r_1^3r_2+(v-u-2x)r_2^2+r_2^3-xr_1(1+r_2)+r_1^2(v-x+r_2+2r_2^2)+r_1r_2(u+2v+r_2(2+r_2)))r_4+d_5(2ur_1+(-1+r_1^2+r_3^2)r_4)(x+y)) \quad (\text{B139})$$

$$A_4^{3'} = -\frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2\sqrt{r_4}}(2x^2+2u^2r_2+(-1+2u+2v)xr_2+2r_1^4r_2+(u-v+3x)r_2^2+(-1+u+v)r_2^3+r_2^4+r_1^3r_2(1+5r_2)+r_1^2(-v-x+r_2(-1+4u+v+3x+r_2(3+4r_2))))+r_1(2x^2+x(-1+2v+r_2(2+3r_2))+$$

$$r_2(u - 2v + r_2(-2 + 5u + 2v + r_2(3 + r_2)))) \quad (\text{B140})$$

$$A_5^{3'} = \frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (2ur_1 + (-1 + r_1^2 + r_3^2)r_4) \quad (\text{B141})$$

$$A_7^{3'} = -\frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (-2(u + x) + r_1^3 + r_1^2(-1 + 2r_2) + r_1(2x + (-2 + r_2)r_2 - r_3^2) - r_2(2u + r_2 + r_3^2)) \quad (\text{B142})$$

$$A_8^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (d_4(-2(u + x) + r_1^3 + r_1^2(-1 + 2r_2) + r_1(2x + (-2 + r_2)r_2 - r_3^2) - r_2(2u + r_2 + r_3^2)) + d_3(2x + r_1^2 + r_1(-1 + r_2) + r_2 + 2r_2^2)r_4) \quad (\text{B143})$$

$$A_{10}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (d_3(2xr_1 + 2xr_2 - r_1^2 r_2 + r_2^3) + d_4(2x - 2r_1^2 r_2 + r_1(2(v + x) + r_2 - 3r_2^2) + r_2(-2u + 2v + r_2 - r_2^2))) \quad (\text{B144})$$

$$A_{11}^{3'} = \frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (-2x + 2r_1^2 r_2 + r_2(2u - 2v - r_2 + r_2^2) + r_1(-2(v + x) + r_2(-1 + 3r_2))) \quad (\text{B145})$$

$$A_{14}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4} (d_3 r_4^2 + d_4(2(u + x) + 2r_1^2 + 5r_1 r_2 + 3r_2^2 + r_4))}{L_2} \quad (\text{B146})$$

$$A_{15}^{3'} = \frac{\kappa d_5 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} (2(u + x) + (1 + 2r_1 + 3r_2)r_4) \quad (\text{B147})$$

$$A_{25}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (d_4(-2x + 2r_1^2 r_2 + r_2(2(u - v + x) + (-1 + r_2)r_2) + r_1(-2(v + x) + r_2(-1 + 3r_2))) + d_5(-2(u + x) + r_1^3 + r_1^2(2r_2 - 1) + r_1(2x + (r_2 - 2)r_2 - r_3^2) - r_2(r_2 + r_3^2)) + d_3(-2x + (r_1 - r_2)r_2)r_4) \quad (\text{B148})$$

$$A_{29}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} (d_3 r_4^2 + d_4(2(u + x) + 2r_1^2 + 3r_1 r_2 + r_2^2 + r_4)) \quad (\text{B149})$$

and

$$A_1^2 = -\frac{A_1^1}{L_1} L_2 - \frac{8L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (d_3(x - r_1 r_2 + r_2^2)r_4 - d_5(2ur_1 + (r_1^2 + r_3^2 - 1)r_4) + d_4((v + 2x)r_4 + 2r_2^2 r_4 - r_1(x + 2r_2 r_4) + r_2(y - 2u))) \quad (\text{B150})$$

$$A_4^2 = -\frac{A_4^1}{L_1} L_2 + \frac{8d_5 L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} ((v + 2x)r_4 + 2r_2^2 r_4 - r_1(x + 2r_2 r_4) + r_2(y - 2u)) \quad (\text{B151})$$

$$A_5^2 = -\frac{A_5^1}{L_1} L_2 \quad (\text{B152})$$

$$A_7^2 = -\frac{A_7^1}{L_1} L_2 - 8d_5 L_2 m_t^{\frac{5}{2}} \sqrt{r_4} \quad (\text{B153})$$

$$A_8^2 = -\frac{A_8^1}{L_1} L_2 + \frac{8L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (d_3 r_4 + d_4 r_4) \quad (\text{B154})$$

$$A_{10}^2 = -\frac{A_{10}^1}{L_1} L_2 - \frac{8d_4 L_2 m_t^{\frac{5}{2}} r_2}{\sqrt{r_4}} \quad (\text{B155})$$

$$A_{11}^2 = -\frac{A_{11}^1}{L_1}L_2 + \frac{8d_5L_2m_t^{\frac{5}{2}}r_2}{\sqrt{r_4}} \quad (\text{B156})$$

$$A_{14}^2 = -\frac{A_{14}^1}{L_1}L_2 - 8d_4L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B157})$$

$$A_{15}^2 = -\frac{A_{15}^1}{L_1}L_2 + 8d_5L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B158})$$

$$A_{25}^2 = -\frac{A_{25}^1}{L_1}L_2 - \frac{8L_2m_t^{\frac{5}{2}}}{\sqrt{r_4}}(d_4r_2 + d_5r_4) \quad (\text{B159})$$

$$A_{29}^2 = -\frac{A_{29}^1}{L_1}L_2 - 8d_4L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B160})$$

$$A_1^{4'} = \frac{A_1^{3'}}{L_1}L_2 - \frac{2\kappa m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(2x^2d_4 + xr_4(-(d_3 + 2d_4)(r_1 - r_2)) + d_4r_4) + r_4(-(vd_4r_4) + d_3r_2r_4^2 + d_4r_2(u + r_4^2)) \quad (\text{B161})$$

$$A_4^{4'} = \frac{A_4^{3'}}{L_1}L_2 - \frac{2\kappa d_5m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(x(2r_1 - 2r_2 - r_4)r_4 - 2x^2 + r_4(vr_4 - ur_2 - r_2r_4^2)) \quad (\text{B162})$$

$$A_5^{4'} = \frac{A_5^{3'}}{L_1}L_2 \quad (\text{B163})$$

$$A_7^{4'} = \frac{A_7^{3'}}{L_1}L_2 - \frac{2\kappa d_5m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(2(u + x) + r_4^2) \quad (\text{B164})$$

$$A_8^{4'} = \frac{A_8^{3'}}{L_1}L_2 + \frac{2\kappa m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(d_3(r_1 - r_2)r_4 + d_4(2(u + x) + r_4^2)) \quad (\text{B165})$$

$$A_{10}^{4'} = \frac{A_{10}^{3'}}{L_1}L_2 - \frac{2\kappa d_4m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(2x + r_2r_4) \quad (\text{B166})$$

$$A_{11}^{4'} = \frac{A_{11}^{3'}}{L_1}L_2 + \frac{2\kappa d_5m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(2x + r_2r_4) \quad (\text{B167})$$

$$A_{14}^{4'} = \frac{A_{14}^{3'}}{L_1}L_2 + \frac{2\kappa d_4m_t^{\frac{7}{2}}r_4^{\frac{3}{2}}}{L_1} \quad (\text{B168})$$

$$A_{15}^{4'} = \frac{A_{15}^{3'}}{L_1}L_2 - \frac{2\kappa d_5m_t^{\frac{7}{2}}r_4^{\frac{3}{2}}}{L_1} \quad (\text{B169})$$

$$A_{25}^{4'} = \frac{A_{25}^{3'}}{L_1}L_2 - \frac{2\kappa m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(2ud_5 + 2x(d_4 + d_5) + d_4r_2r_4 + d_5r_4^2) \quad (\text{B170})$$

$$A_{29}^{4'} = \frac{A_{29}^{3'}}{L_1}L_2 + \frac{2\kappa d_4m_t^{\frac{7}{2}}r_4^{\frac{3}{2}}}{L_1}. \quad (\text{B171})$$

The remaining coefficients for the case of $[^3P_0]_1$:

$$A_2^1 = \frac{L_1m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(d_2(1 + r_2)(r_1^2 - r_1^3 + r_2(u + r_2) + r_1(-u + r_2(2 + r_2))) + d_1(r_1^4 + 2xr_1(1 + r_2) + r_1^3(7r_2 - 1) + r_1r_2(6u + (r_2 - 7)r_2) + r_1^2(u + r_2(9r_2 - 5)) - r_2(2x(1 + r_2) + r_2(r_2(3 + 2r_2) - u))) + 2d_5r_1r_2(1 + r_2)(-2ur_1 -$$

$$(r_1^2 - 1 + r_3^2)r_4)) \quad (\text{B172})$$

$$\begin{aligned} A_3^1 = & -\frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (d_2(-r_1^4 r_2 + r_1^3(r_2 + v + x + z) + r_1 r_2(2x + r_2(2(u + v + 2x) + \\ & r_2(3r_2 - 1))) + r_1^2(v + r_2(3r_2 - u - v - w - 3x + 5z + 2) - r_2^2(v + 2x + \\ & 3r_2^2 - r_2(-2 + v + r_3^2))) + d_1(r_1^3 r_2 + r_2^2(r_2 - 3v + 4z) + r_1^2(v + 2x + \\ & r_2 + 6r_2^2) + r_1 r_2(6r_2 - 7v - 3x + 13z))r_4 + 2d_5 r_1 r_2(2(v + x) + \\ & r_2(2 + r_1 + 3r_2))(2ur_1 + (-1 + r_1^2 + r_3^2)r_4)) \end{aligned} \quad (\text{B173})$$

$$A_6^1 = \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (-d_2(r_1 - r_2)(1 + r_2) + d_1(r_1^2 + 6r_1 r_2 + r_2^2)) \quad (\text{B174})$$

$$A_{24}^1 = 0 \quad (\text{B175})$$

$$A_{26}^1 = \frac{4d_5 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (r_2 - v - x + 3z) \quad (\text{B176})$$

$$A_{27}^1 = \frac{4L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (d_4(1 + r_1)(1 + r_2) + d_3(1 + r_1 + 2r_2)r_4) \quad (\text{B177})$$

$$A_{28}^1 = 8d_3 L_1 m_t^{\frac{5}{2}} \sqrt{r_4} (1 + r_2) \quad (\text{B178})$$

$$A_{30}^1 = 4d_5 L_1 m_t^{\frac{5}{2}} \sqrt{r_4} (1 + r_2) \quad (\text{B179})$$

$$\begin{aligned} A_2^{3'} = & \frac{-\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 \sqrt{r_4}} (2d_5 r_1 r_2(2ur_1 + (r_1^2 - 1 + r_3^2)r_4)(x + y) + d_1(4x^2(r_2 - r_1) \\ & + (r_1^3 + r_1^2(1 + 6r_2) + r_2(3u + r_2 + 4r_2^2) + r_1(u + r_2(6 + 13r_2)))r_4^2 + \\ & 2x(u(r_2 - r_1) + 4r_2 r_4^2)) + d_2(r_2(-4u(u + x) - (u + 2x)(2r_1 - 1)r_4 - 2ur_4^2 + \\ & 2r_3^2 r_4^2 + (1 - 2r_1)r_4^3) + r_4(-r_3^2 r_4^2 + r_1(1 + r_4)(x + y) + u(u + x + y)))) \end{aligned} \quad (\text{B180})$$

$$\begin{aligned} A_3^{3'} = & \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} (2xd_1(r_1^2 - r_1(v + (-5 + r_2)r_2) + r_2(v + r_2(2 + r_2)))r_4 + \\ & d_1(v(r_1 + 3r_2) + r_2(r_1 - r_1^2 - 5r_1 r_2 + r_2(3 + 2r_2)))r_4^3 + 2d_5 r_1 r_2(2ur_1 + \\ & (-1 + r_1^2 + r_3^2)r_4)(2x + vr_4 + r_2^2 r_4 + r_2(r_4 - 2y)) + d_2(4x^2 r_1(r_1 - r_2) + \\ & 2x(r_1^3 + r_1^4 - r_2^2(v - 1 + r_2) + r_1^2(-1 + 2u + v - 3(-1 + r_2)r_2) + \\ & r_1 r_2(-2u + r_2 - 2r_2^2)) + v(r_1^2(1 + r_2) + 2r_1(u + r_2) + \\ & r_2(-2u + r_2 - r_2^2))r_4 + r_2 r_4(-2u(r_2 + r_1(-1 + r_4)) + r_4(r_1^2 - 2r_1^3 + \\ & 2r_1 r_2 - 5r_1^2 r_2 - r_2^2 - r_1 r_2^2 + r_4)))) \end{aligned} \quad (\text{B181})$$

$$\begin{aligned} A_6^{3'} = & \frac{-(\kappa m_t^{\frac{7}{2}})}{4L_2 r_1 r_2 \sqrt{r_4}} (-2xd_1(r_1 - r_2) + d_2(2r_1^3 + r_1^2(r_2 - 1) - 2r_1(-u - x + r_2 + r_2^2) \\ & - r_2(r_2 + 2u - v + x + z)) + d_1(r_1 + 3r_2)r_4^2 + 2d_5 r_1 r_2(2ur_1 + (r_1^2 - 1 + r_3^2)r_4)) \end{aligned} \quad (\text{B182})$$

$$A_{24}^{3'} = \frac{2\kappa d_3 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} \quad (\text{B183})$$

$$A_{26}^{3'} = -\frac{\kappa d_5 m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (2(x + (v + x)r_1 + vr_2) + r_2^2 r_4 + r_2(-2(u + x) + r_4 - 2r_4^2)) \quad (\text{B184})$$

$$A_{27}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}}}{L_2 \sqrt{r_4}} (-(d_3(-2v + 2x - r_2 + r_1(-1 + r_4))r_4) + d_4(2(u + x) - 2xr_1 + r_4(r_3^2 + r_4 - r_1 r_4))) \quad (\text{B185})$$

$$A_{28}^{3'} = \frac{2\kappa d_3 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} (x + y) \quad (\text{B186})$$

$$A_{30}^{3'} = \frac{\kappa d_5 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} (2(u + x) + (1 + 2r_1 + r_2)r_4) \quad (\text{B187})$$

and

$$A_2^2 = -\frac{A_2^1}{L_1} L_2 - \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (-2xd_1(-2r_2 + r_4) + 2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + d_1 r_4 (2r_2 r_4 + r_4^2) + d_2(-2ur_2 + ur_4 - r_2^2 r_4 + r_1 r_4(-3r_2 + r_4))) \quad (\text{B188})$$

$$A_3^2 = -\frac{A_3^1}{L_1} L_2 + \frac{2L_2 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (2xd_2(r_1 - r_4)(-2r_2 + r_4) + d_2 r_4 (v(2r_2 - r_4) + (r_1 - r_2)r_2(-2r_1 - 4r_2 + r_4)) + r_2(-(d_1 r_4 (2(r_1 - r_2)r_2 + (r_1 + 3r_2)r_4)) + 4d_5 r_1 r_2 (-2ur_1 - (-1 + r_1^2 + r_3^2)r_4))) \quad (\text{B189})$$

$$A_6^2 = -\frac{A_6^1}{L_1} L_2 - \frac{2d_2 L_2 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (-2r_2 + r_4) \quad (\text{B190})$$

$$A_{24}^2 = -\frac{A_{24}^1}{L_1} L_2 \quad (\text{B191})$$

$$A_{26}^2 = -\frac{A_{26}^1}{L_1} L_2 + \frac{8d_5 L_2 m_t^{\frac{5}{2}} r_2}{\sqrt{r_4}} \quad (\text{B192})$$

$$A_{27}^2 = -\frac{A_{27}^1}{L_1} L_2 + \frac{8L_2 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (d_3 r_4 + d_4 r_4) \quad (\text{B193})$$

$$A_{28}^2 = -\frac{A_{28}^1}{L_1} L_2 + 16d_3 L_2 m_t^{\frac{5}{2}} \sqrt{r_4} \quad (\text{B194})$$

$$A_{30}^2 = -\frac{A_{30}^1}{L_1} L_2 + 8d_5 L_2 m_t^{\frac{5}{2}} \sqrt{r_4} \quad (\text{B195})$$

$$A_2^{4'} = \frac{A_2^{3'}}{L_1} L_2 + \frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (d_1 r_4 (2(r_1 - r_2)r_2 + (r_1 + 3r_2)r_4) + d_2 r_4 (x + y)) \quad (\text{B196})$$

$$A_3^{4'} = \frac{A_3^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{2L_1 r_1 r_2 r_4^{\frac{3}{2}}} (2d_5 r_1 r_2 (2x + r_2 r_4) (2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + d_1 r_4 (r_2 r_4 (2r_2 r_4 + r_4^2) + 2x(2r_2 r_4 + r_1(2r_2 + r_4))) + d_2 r_4 (2xr_1^2 + 2xr_1 r_2 + r_1 r_4 (v + 3r_2^2 - r_2 r_4) + r_2 (r_2^2 r_4 - 4yr_2 + r_4(2u + v + 2x + 2r_4^2)))) \quad (\text{B197})$$

$$A_6^{4'} = \frac{A_6^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{7}{2}} r_4^{\frac{3}{2}}}{2L_1 r_1 r_2} \quad (\text{B198})$$

$$A_{24}^{4'} = \frac{A_{24}^{3'}}{L_1} L_2 \quad (\text{B199})$$

$$A_{26}^{4'} = \frac{A_{26}^{3'}}{L_1} L_2 + \frac{2\kappa d_5 m_t^{\frac{7}{2}}}{L_1 \sqrt{r_4}} (2x + r_2 r_4) \quad (\text{B200})$$

$$A_{27}^{4'} = \frac{A_{27}^{3'}}{L_1} L_2 + \frac{2\kappa m_t^{\frac{7}{2}}}{L_1 \sqrt{r_4}} (d_3 r_4^2 + d_4 (u + x + y)) \quad (\text{B201})$$

$$A_{30}^{4'} = \frac{A_{30}^{3'}}{L_1} L_2 - \frac{2\kappa d_5 m_t^{\frac{7}{2}} r_4^{\frac{3}{2}}}{L_1}. \quad (\text{B202})$$

The remaining coefficients for the case of $[^3P_1]_1$ state:

$$A_1^1 = \frac{-2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_2 (1 + r_2) (u + r_2 + r_1 (-1 + r_4)) r_4 - d_1 r_4^2 (u + (-1 + r_1) r_4) + 2r_1 r_2 (- (d_4 (-x + 2u(v + x) + (u - v - 3x) r_2 + r_1^3 r_2 - 3r_2^3 + r_1^2 (2(v + x) + r_2 + 4r_2^2) + r_2^2 (2u + w - 2) + r_1 (-v - x + r_2 (u - 2r_2 - v + w + 3z - 2) + r_3^2))) + d_3 (x + r_1^2 r_2 + r_1 (-v - x + (r_2 - 1) r_2) + r_2 (r_2 + u - v + 2z)) r_4)) \quad (\text{B203})$$

$$A_4^1 = \frac{2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (d_2 (-x + r_1 (r_2 + v + x + z) - r_2 (r_2 - v + 3z)) r_4 + x (2d_5 r_1 r_2 (2u + (r_1 - 1) (1 + 2r_1 + 3r_2)) + d_1 r_4^2) + r_2 (d_1 r_4^3 + 2d_5 r_1 (u r_2 (1 + 2r_1 + 3r_2) + v (2u + r_1 (2r_1 - 1) - r_2 + 3r_1 r_2 + r_2^2) + r_2 ((r_1 - 1) (2 + r_1 + 3r_2) + r_3^2) r_4)) \quad (\text{B204})$$

$$A_5^1 = \frac{2L_1 m_t^{\frac{5}{2}} \sqrt{r_4}}{r_1 r_2} (d_2 - d_1 r_1 + (d_1 + d_2) r_2) \quad (\text{B205})$$

$$A_7^1 = \frac{2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (4ud_5 r_1 (r_1 - r_2) r_2 + r_4 (2d_5 r_1 (1 + r_1) (-3 + 2r_1 - r_2) r_2 + d_2 (2r_1^2 - r_1 (-3 + r_2) + r_2 (3 + r_2)) + d_1 (r_1 - r_2) r_4)) \quad (\text{B206})$$

$$A_8^1 = \frac{4L_1 m_t^{\frac{5}{2}}}{r_1 r_2 \sqrt{r_4}} (r_1 r_2 (d_3 (1 + r_1) r_4 + d_4 (1 + 2u + r_1 r_2 + 2r_3^2 + r_4)) - d_1 (1 + r_2) r_4) \quad (\text{B207})$$

$$A_9^1 = \frac{-2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (d_2 (r_1 (2 + r_1) + r_2) r_4 + 2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2) r_4)) \quad (\text{B208})$$

$$A_{10}^1 = \frac{-2L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (-d_2 (r_1 (-3 + r_2) - 3r_2 (1 + r_2)) r_4 + d_1 r_4^3 + 2r_1 r_2 (4ud_5 r_1 + 2d_5 (-1 + r_1^2 + r_3^2) r_4 + r_4 (d_4 (r_2 + v + x + 3z) - d_3 r_2 r_4))) \quad (\text{B209})$$

$$A_{11}^1 = \frac{4d_5 L_1 m_t^{\frac{5}{2}}}{\sqrt{r_4}} (r_2 + v + x + 3z) \quad (\text{B210})$$

$$A_{12}^1 = \frac{2d_2 L_1 m_t^{\frac{5}{2}} r_2}{r_1 \sqrt{r_4}} \quad (\text{B211})$$

$$A_{13}^1 = \frac{L_1 m_t^{\frac{5}{2}}}{r_1 r_2 r_4^{\frac{3}{2}}} (8ud_5 r_1^2 r_2 + r_4 (d_2 (- (r_1 (-3 + r_2)) + 3r_2 (1 + r_2)) +$$

$$4d_5r_1r_2(-1 + r_1^2 + r_3^2) + d_1(r_1 - r_2)r_4)) \quad (\text{B212})$$

$$A_{14}^1 = \frac{4L_1m_t^{\frac{5}{2}}}{\sqrt{r_4}}(2ud_4 + (d_3(-r_1 + r_2) + d_4(-1 + 2r_1 + r_2))r_4) \quad (\text{B213})$$

$$A_{15}^1 = \frac{4L_1m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(-2ud_5r_1r_2 - (d_2(r_1 - r_2) + d_5r_1r_2(-1 + 2r_1 + r_2))r_4) \quad (\text{B214})$$

$$A_{16}^1 = \frac{L_1m_t^{\frac{5}{2}}\sqrt{r_4}}{r_1r_2}(d_2 - d_1r_1 + 2d_2r_1 + d_1r_2 - d_2r_2) \quad (\text{B215})$$

$$A_{17}^1 = -\frac{L_1m_t^{\frac{5}{2}}}{r_1r_2r_4^{\frac{3}{2}}}(d_2(-r_1^3r_2 + r_1^2(4(v + x) + r_2) + r_2(v + 4x + 3r_2^2 - r_2(-2 + 2u + 3v + r_3^2))) + r_1(v + 4x - r_2(u - v + w + x + 3z - 2)))r_4 + \\ d_1(r_1^2r_2 - r_2(v + r_2) + r_1(r_2 - 2v - x + 3z))r_4^2 + 2d_5r_1r_2(2(v + 2x) + \\ r_2(2 + r_1 + 3r_2))(2ur_1 + (-1 + r_1^2 + r_3^2)r_4)) \quad (\text{B216})$$

$$A_{18}^1 = \frac{L_1m_t^{\frac{5}{2}}}{r_1r_2r_4^{\frac{3}{2}}}(-2d_5r_1r_2(-2u - (-1 + 2r_1 + r_2)r_4)(2ur_1 + (-1 + r_1^2 + r_3^2)r_4) + \\ d_1r_4^2(u(r_1 - r_2) - 2x(1 + r_2) + (r_1 - 1 - 2r_2)r_4^2) + d_2r_4(u(4r_1^2 - r_1(r_2 - 3) \\ -(r_2 - 3)r_2) + r_4((r_1 - r_2)(1 + r_2 + 2r_3^2) + r_1(1 + 2r_1 - r_2)r_4 + 2r_4^2))) \quad (\text{B217})$$

$$A_{19}^1 = \frac{L_1m_t^{\frac{5}{2}}}{r_1r_2r_4^{\frac{3}{2}}}(d_2(xr_1(-3 + r_2) - 2r_1r_2^3 - r_1^2(r_2 + v + x + 3z) + r_2(-3x + \\ r_2(r_2 + 3v + z)))r_4 - d_1r_4^3(x + r_2r_4) - 2d_5r_1r_2(2x + r_2r_4)(2ur_1 + \\ (-1 + r_1^2 + r_3^2)r_4)) \quad (\text{B218})$$

$$A_{25}^1 = \frac{-2L_1m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(d_2(1 + r_2)r_4 - d_1r_4^2 + 2r_1r_2(d_5(1 + r_1)(1 + r_2) - \\ d_4(r_2 - v - x + 3z) + d_3r_2r_4)) \quad (\text{B219})$$

$$A_{29}^1 = -4L_1m_t^{\frac{5}{2}}\sqrt{r_4}(d_4 + d_3r_1 - d_3r_2 + d_4r_2) \quad (\text{B220})$$

$$A_1^{3'} = \frac{-\kappa m_t^{\frac{7}{2}}}{2L_2r_1r_2\sqrt{r_4}}(2d_4r_1r_2(-2x(x + (u + x)r_1) + x(2 - 2v + 2r_1 - r_3^2)r_4 + \\ (v - x)(1 + r_1)r_4^2 - vr_4^3 + r_2^2r_4(u + r_4 + r_1r_4) + r_2(u + r_4 + r_1r_4) \cdot \\ (-2(u + x) + r_4 - 2r_4^2)) + r_4(2xd_1(-u + r_4^2) + d_1r_4^2(u + 2v + r_1 + \\ r_2 + r_1r_2 + r_2^2 + r_4^2) + d_2(2u^2 + 2ux + ur_4^2 - r_3^2r_4^2 - r_2(x + y) + \\ r_1(1 + r_4)(x + y)) - 2d_3r_1r_2(-(x(2(u + x) + r_1)) - (u + x)r_2^2 + \\ (v - x)r_4^2 + r_2(x + (u + x)r_1 + (1 + r_1)r_4^2)))) \quad (\text{B221})$$

$$A_4^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2r_1r_2\sqrt{r_4}}(2d_5r_1r_2(2x(x + (u + x)r_1) + x(-2 + 2v - 2r_1 + r_3^2)r_4 - \\ (v - x)(1 + r_1)r_4^2 + vr_4^3 - r_2^2r_4(u + r_4 + r_1r_4) + r_2(u + r_4 + r_1r_4) \cdot$$

$$(2(u+x) + r_4(-1+2r_4))) + r_4(d_1(2x^2 + xr_4^2 + (r_1-r_2)r_2r_4^2) + d_2(-(x(2(u+x) + r_1)) + xr_1r_4 + (2v-x)r_4^2 - r_2^2(2(u+x) + r_4^2) + r_2(x + 2(u+x)r_1 + (1+2r_1)r_4^2)))) \quad (\text{B222})$$

$$A_5^{3'} = \frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_2 r_1 r_2} (d_2(2(u+x) + r_1(2r_1-1) + r_2 + 3r_1r_2 + r_2^2) + d_1(2x + r_4^2)) \quad (\text{B223})$$

$$A_7^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 \sqrt{r_4}} (-(d_1r_4(2x + r_4^2)) + d_2r_4(2(u+x) + r_2 + r_1(-1+r_4) + r_4^2) - 2d_5r_1r_2(2x(r_1-1) + 2u(r_1-1-r_2) + (r_1^2-1)r_4 + (r_1-1)r_4^2)) \quad (\text{B224})$$

$$A_8^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 \sqrt{r_4}} (2d_3r_1r_2(2x + r_1^2 + r_1(r_2-1) + r_2 + 2r_2^2)r_4 - 2d_1r_4(2x + r_4^2) - 2d_4r_1r_2(-2xr_1 + 2(u+x+ur_2) + r_4(r_3^2 + r_4 - r_1r_4))) \quad (\text{B225})$$

$$A_9^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 r_4^{\frac{3}{2}}} (2d_5r_1r_2(1+r_2)(2ur_1 + (r_1^2-1+r_3^2)r_4) - d_2r_4(-(r_1^2(1+r_2)) + (1+r_2-r_3^2)r_4 + r_1(2x+r_2+r_2^2+r_4^2))) \quad (\text{B226})$$

$$A_{10}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 \sqrt{r_4}} (4xd_4r_1r_2 + 4xd_4r_1^2r_2 - 4ud_4r_1r_2^2 - 2xd_1r_4 + 4xd_3r_1r_2r_4 + 4vd_4r_1r_2r_4 + 2d_4r_1r_2^2r_4 - 2d_3r_1^2r_2^2r_4 + 2d_3r_1r_2^3r_4 + 2d_4r_1r_2^3r_4 - 4d_4r_1r_2^2r_4^2 + d_1r_4^3 + 2d_5r_1r_2(2ur_1 + (-1+r_1^2+r_3^2)r_4) - d_2r_4(2(u+x) - r_2(1+2r_2) + r_4^2 + r_1(1+2r_2+r_4))) \quad (\text{B227})$$

$$A_{11}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}}}{L_2 r_1 \sqrt{r_4}} (d_2(r_1-r_2)r_4 + d_5r_1(2x(1+r_1) - 2ur_2 + 2vr_4 + r_2(1+r_2-2r_4)r_4)) \quad (\text{B228})$$

$$A_{12}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{2L_2 r_1 r_2 r_4^{\frac{3}{2}}} (d_2(r_1^2r_2 - 2r_1(z+r_2) - r_2(2v+r_2(2+r_2)))r_4 + 2d_5r_1r_2^2(2ur_1 + (-1+r_1^2+r_3^2)r_4)) \quad (\text{B229})$$

$$A_{13}^{3'} = \frac{-(\kappa m_t^{\frac{7}{2}})}{4L_2 r_1 r_2 \sqrt{r_4}} (d_2(-2(u+x) - r_1(1+2r_1) + r_2 - 5r_1r_2 + r_2^2)r_4 + 2d_5r_1r_2(2ur_1 + (-1+r_1^2+r_3^2)r_4) + d_1r_4(2x+r_4^2)) \quad (\text{B230})$$

$$A_{14}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2 r_1 r_2} (r_1r_2(d_3r_4^2 + d_4(2(u+x) + 2r_1^2 + 5r_1r_2 + 3r_2^2 + r_4)) - d_1r_4^2) \quad (\text{B231})$$

$$A_{15}^{3'} = \frac{\kappa d_5 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_2} (2(u+x) + (1+2r_1+3r_2)r_4) \quad (\text{B232})$$

$$A_{16}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 \sqrt{r_4}} (d_2(2(u+x) + 2r_1^2 + r_1(-1+r_2) + r_2 + 3r_2^2)r_4 + 2d_5r_1r_2(2ur_1 + (-1+r_1^2+r_3^2)r_4) + d_1r_4(2x+r_4^2)) \quad (\text{B233})$$

$$A_{17}^{3'} = \frac{\kappa m_t^{\frac{7}{2}}}{4L_2 r_1 r_2 r_4^{\frac{3}{2}}} (2xd_1(v+r_1+r_2^2)r_4^2 + d_1(v+r_2-r_1r_2+2r_2^2)r_4^4 +$$

$$\begin{aligned}
& 2d_5r_1r_2(2ur_1 + (-1 + r_1^2 + r_3^2)r_4)(4x + r_4(v + r_2 + r_2^2 - 2r_2r_4)) + \\
& d_2r_4(v(-2u + 3r_1(1 + r_2) - r_2(3 + r_2))r_4 - r_2(2u + 2r_1^3 + r_1^2(r_2 - 1) + \\
& r_2 - r_1(1 + r_2(4 + r_2) + 2r_3^2) + r_2(-r_2 + 2(u + r_3^2)))r_4 + \\
& 2x(2r_1(r_1 - r_2) + (-2 + v + r_3^2)r_4))) \tag{B234}
\end{aligned}$$

$$\begin{aligned}
A_{18}^{3'} &= \frac{-\kappa m_t^{\frac{7}{2}}}{4L_2r_1r_2\sqrt{r_4}}(2d_5r_1r_2(u + 2x + r_1(2 + r_1) + 2r_2 + 4r_1r_2 + 3r_2^2)(2ur_1 + \\
& (-1 + r_1^2 + r_3^2)r_4) + d_2r_4(-2u^2 - u(2r_1^2 + r_1(-3 + r_2) + 3r_2(1 + r_2)) - \\
& 2x(u + r_2 + r_1(r_4 - 1)) - ((r_1 - 2)(r_1 - 1)r_1 + 2r_2 + 4r_1r_2 + 3(1 + r_1)r_2^2)r_4 \\
& + r_3^2r_4^2) + d_1r_4(4x^2 + (u + r_1 + r_1^2 - r_2 + 3r_1r_2)r_4^2 + 2x(u + 2r_4^2))) \tag{B235}
\end{aligned}$$

$$\begin{aligned}
A_{19}^{3'} &= \frac{-(\kappa m_t^{\frac{7}{2}})}{4L_2r_1r_2\sqrt{r_4}}(d_2(2x(u + x) + xr_2 + 4r_1^3r_2 - (4u + 2v + x)r_2^2 - 3r_2^3 - 3r_2^4 + \\
& r_1^2(-2v + (-3 + r_2)r_2) - r_1(x + r_2(6r_2 - 4u - 2v - 9x + 6z)))r_4 + d_1(2x + \\
& r_1^2 - r_2^2)r_4(x + r_2r_4) + 2d_5r_1r_2(x + 2r_2r_4)(2ur_1 + (r_1^2 - 1 + r_3^2)r_4))) \tag{B236}
\end{aligned}$$

$$\begin{aligned}
A_{25}^{3'} &= \frac{-(\kappa m_t^{\frac{7}{2}})}{2L_2r_1r_2\sqrt{r_4}}(-4ud_4r_1r_2^2 + 4vd_4r_1r_2r_4 + 2d_4r_1r_2^2r_4 - 2d_3r_1^2r_2^2r_4 + \\
& 2d_3r_1r_2^3r_4 + 2d_4r_1r_2^3r_4 - 4d_4r_1r_2^2r_4^2 + d_1r_4^3 + d_2r_4(2(u + x) + r_2 + \\
& r_1(-1 + r_4) + r_4^2) + 2d_5(-1 + r_1)r_1r_2(2(u + x) + r_4(1 + r_1 + r_4)) + \\
& 2x(-d_1r_4 + 2r_1r_2(d_4 + d_4r_1 - d_4r_2 + d_3r_4))) \tag{B237}
\end{aligned}$$

$$A_{29}^{3'} = -\frac{\kappa m_t^{\frac{7}{2}}\sqrt{r_4}}{L_2r_1r_2}(r_1r_2(d_3r_4^2 + d_4(2(u + x) + 2r_1^2 + 3r_1r_2 + r_2^2 + r_4) - d_1r_4^2)) \tag{B238}$$

and

$$\begin{aligned}
A_1^2 &= -\frac{A_1^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(2r_1r_2(d_3(xr_1 + xr_2 - r_1^2r_2 + r_2^3) + d_4(-r_1^2r_2 + \\
& r_1(z + r_2^2) + r_2(3z - u - 2v))) + d_2(u + r_1^2 + r_2^2)r_4 + d_1r_4^3) \tag{B239}
\end{aligned}$$

$$\begin{aligned}
A_4^2 &= -\frac{A_4^1}{L_1}L_2 + \frac{4L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(d_2(-x + (r_1 - r_2)r_2)r_4 + \\
& 2d_5r_1r_2(-(v + 2x)r_4 - 2r_2^2r_4 + r_1(x + 2r_2r_4) - r_2(-u + x + r_4^2))) \tag{B240}
\end{aligned}$$

$$A_5^2 = -\frac{A_5^1}{L_1}L_2 + \frac{4d_2L_2m_t^{\frac{5}{2}}\sqrt{r_4}}{r_1r_2} \tag{B241}$$

$$A_7^2 = -\frac{A_7^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(-3d_2r_4 + 2d_5r_1r_2r_4) \tag{B242}$$

$$A_8^2 = -\frac{A_8^1}{L_1}L_2 + \frac{8L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(-(d_1r_4) + d_3r_1r_2r_4 + d_4r_1r_2r_4) \tag{B243}$$

$$A_9^2 = -\frac{A_9^1}{L_1}L_2 - \frac{4d_2L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(r_1 + r_4) \tag{B244}$$

$$A_{10}^2 = -\frac{A_{10}^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(2d_4r_1r_2^2 + 3d_2r_4) \quad (\text{B245})$$

$$A_{11}^2 = -\frac{A_{11}^1}{L_1}L_2 + \frac{8d_5L_2m_t^{\frac{5}{2}}r_2}{\sqrt{r_4}} \quad (\text{B246})$$

$$A_{12}^2 = -\frac{A_{12}^1}{L_1}L_2 \quad (\text{B247})$$

$$A_{13}^2 = -\frac{A_{13}^1}{L_1}L_2 + \frac{6d_2L_2m_t^{\frac{5}{2}}\sqrt{r_4}}{r_1r_2} \quad (\text{B248})$$

$$A_{14}^2 = -\frac{A_{14}^1}{L_1}L_2 - 8d_4L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B249})$$

$$A_{15}^2 = -\frac{A_{15}^1}{L_1}L_2 + 8d_5L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B250})$$

$$A_{16}^2 = -\frac{A_{16}^1}{L_1}L_2 + \frac{2d_2L_2m_t^{\frac{5}{2}}\sqrt{r_4}}{r_1r_2} \quad (\text{B251})$$

$$A_{17}^2 = -\frac{A_{17}^1}{L_1}L_2 - \frac{2L_2m_t^{\frac{5}{2}}}{r_1r_2r_4^{\frac{3}{2}}}(d_2r_4(2r_1^2r_2 + (v + 4x + 3r_2^2)r_4 - r_1r_2(2r_2 + r_4)) + r_2(d_1(r_1 - r_2)r_4^2 + 4d_5r_1r_2(2ur_1 + (-1 + r_1^2 + r_3^2)r_4))) \quad (\text{B252})$$

$$A_{18}^2 = -\frac{A_{18}^1}{L_1}L_2 + \frac{2L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(-2d_5r_1r_2(2ur_1 + (-1 + r_1^2 + r_3^2)r_4) - d_1r_4(2x + r_4^2) + d_2r_4(3u - r_2^2 + 2r_4^2 + r_1(r_2 + r_4))) \quad (\text{B253})$$

$$A_{19}^2 = -\frac{A_{19}^1}{L_1}L_2 - \frac{2d_2L_2m_t^{\frac{5}{2}}\sqrt{r_4}}{r_1r_2}(3x + (r_1 - r_2)r_2) \quad (\text{B254})$$

$$A_{25}^2 = -\frac{A_{25}^1}{L_1}L_2 - \frac{4L_2m_t^{\frac{5}{2}}}{r_1r_2\sqrt{r_4}}(d_2r_4 + 2r_1r_2(-(d_4r_2) + d_5r_4)) \quad (\text{B255})$$

$$A_{29}^2 = -\frac{A_{29}^1}{L_1}L_2 - 8d_4L_2m_t^{\frac{5}{2}}\sqrt{r_4} \quad (\text{B256})$$

$$A_1^{4'} = \frac{A_1^{3'}}{L_1}L_2 + \frac{\kappa m_t^{\frac{7}{2}}}{L_1r_1r_2\sqrt{r_4}}(2d_4r_1r_2(-2x^2 - xr_4(-2r_1 + 2r_2 + r_4) + r_4(-ur_2 + vr_4 - r_2r_4^2)) + r_4(d_1r_4^3 + d_2(r_1 - r_2)(x + y) + 2d_3r_1r_2(xr_1 - r_2(x + r_4^2)))) \quad (\text{B257})$$

$$A_4^{4'} = \frac{A_4^{3'}}{L_1}L_2 + \frac{\kappa m_t^{\frac{7}{2}}}{L_1r_1r_2\sqrt{r_4}}(-d_2r_4(-xr_1 + r_2(x + r_4^2)) + 2d_5r_1r_2(-2x^2 + x(2r_1 - 2r_2 - r_4)r_4 + r_4(-ur_2 + vr_4 - r_2r_4^2))) \quad (\text{B258})$$

$$A_5^{4'} = \frac{A_5^{3'}}{L_1}L_2 - \frac{\kappa d_2m_t^{\frac{7}{2}}\sqrt{r_4}}{L_1r_1r_2}(-r_1 + r_2) \quad (\text{B259})$$

$$A_7^{4'} = \frac{A_7^{3'}}{L_1}L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{L_1r_1r_2\sqrt{r_4}}(d_2(r_2 - r_1)r_4 + 2d_5r_1r_2(2(u + x) + r_4^2)) \quad (\text{B260})$$

$$A_8^{4'} = \frac{A_8^{3'}}{L_1}L_2 + \frac{2\kappa m_t^{\frac{7}{2}}}{L_1\sqrt{r_4}}(d_3(r_1 - r_2)r_4 + d_4(u + x + y)) \quad (\text{B261})$$

$$A_9^{4'} = \frac{A_9^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{L_1 r_1 r_2 r_4^{\frac{3}{2}}} (d_2 r_4 (r_1^2 - r_1 r_2 - r_2 r_4) + 2d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2) r_4)) \quad (\text{B262})$$

$$A_{10}^{4'} = \frac{A_{10}^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}} (d_2 (-r_1 + r_2) r_4 + 2d_4 r_1 r_2 (2x + r_2 r_4))}{L_1 r_1 r_2 \sqrt{r_4}} \quad (\text{B263})$$

$$A_{11}^{4'} = \frac{A_{11}^{3'}}{L_1} L_2 + \frac{2\kappa d_5 m_t^{\frac{7}{2}}}{L_1 \sqrt{r_4}} (2x + r_2 r_4) \quad (\text{B264})$$

$$A_{12}^{4'} = \frac{A_{12}^{3'}}{L_1} L_2 + \frac{2\kappa d_2 m_t^{\frac{7}{2}} \sqrt{r_4}}{L_1 r_1} \quad (\text{B265})$$

$$A_{13}^{4'} = \frac{A_{13}^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (r_1 - r_2) \quad (\text{B266})$$

$$A_{14}^{4'} = \frac{A_{14}^{3'}}{L_1} L_2 + \frac{2\kappa d_4 m_t^{\frac{7}{2}} r_4^{\frac{3}{2}}}{L_1} \quad (\text{B267})$$

$$A_{15}^{4'} = \frac{A_{15}^{3'}}{L_1} L_2 - \frac{2\kappa d_5 m_t^{\frac{7}{2}} r_4^{\frac{3}{2}}}{L_1} \quad (\text{B268})$$

$$A_{16}^{4'} = \frac{A_{16}^{3'}}{L_1} L_2 + \frac{\kappa d_2 m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (r_1 - r_2) \quad (\text{B269})$$

$$A_{17}^{4'} = \frac{A_{17}^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{2L_1 r_1 r_2 r_4^{\frac{3}{2}}} (2d_5 r_1 r_2 (4x + r_2 r_4) (2ur_1 + (-1 + r_1^2 + r_3^2) r_4) + d_1 r_4^2 (2xr_1 + r_2 r_4^2) + d_2 r_4 (4xr_1 (r_1 - r_2) + r_1 r_4 (3v + r_2^2 - r_2 r_4) - r_2 r_4 (4u + 2v + x - 2y + z))) \quad (\text{B270})$$

$$A_{18}^{4'} = \frac{A_{18}^{3'}}{L_1} L_2 + \frac{\kappa m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (d_1 (r_1 - r_2) r_4^2 + 4d_5 r_1 r_2 (2ur_1 + (-1 + r_1^2 + r_3^2) r_4) + d_2 (-2xr_2 + 2r_1^2 r_4 - 3r_2 (u + r_4^2) + r_1 (3u + 2x - 2r_2 r_4 + r_4^2))) \quad (\text{B271})$$

$$A_{19}^{4'} = \frac{A_{19}^{3'}}{L_1} L_2 - \frac{\kappa d_2 m_t^{\frac{7}{2}} \sqrt{r_4}}{2L_1 r_1 r_2} (xr_1 - xr_2 + 3r_2 r_4^2) \quad (\text{B272})$$

$$A_{25}^{4'} = \frac{A_{25}^{3'}}{L_1} L_2 - \frac{\kappa m_t^{\frac{7}{2}}}{L_1 r_1 r_2 \sqrt{r_4}} (d_2 (r_1 - r_2) r_4 + 2r_1 r_2 (d_5 (u + x + y) - d_4 (2x + r_2 r_4))) \quad (\text{B273})$$

$$A_{29}^{4'} = \frac{A_{29}^{3'}}{L_1} L_2 + \frac{2\kappa d_4 m_t^{\frac{7}{2}} r_4^{\frac{3}{2}}}{L_1}. \quad (\text{B274})$$

Here for convenience an overall factor \mathcal{C}_s has been contracted out from these coefficients. Then the square of the amplitude $|M_i|^2$ can be conveniently obtained with the help of

Eqs.(B26,B27,B28) and Eqs.(34, 35,36).

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